

Functions

A function is a special kind of relation. More precisely...

A function f from A to B is a relation on $A \times B$ such that

for all $x \in A$, there is exactly one $y \in B$ s.t. $(x, y) \in f$.

The set A is called the *domain* of f , and B the *codomain* of f .

We say that a function f from A to B has *type* $A \rightarrow B$,
and we write

$$f : A \rightarrow B.$$

In such a function f , there is a unique $y \in B$ for each $x \in A$,
so the standard functional notation

$$f(x) = y$$

is well-defined and convenient.

We call x the *argument* of f , and y is called the *value* of f at x .

If A and B are nonempty finite sets, how many functions of type $A \rightarrow B$?

Multi-argument functions

It may seem that we have given a definition of a function that is suitable only for unary functions (that is, functions that take one argument).

But there is an easy, standard way to incorporate multi-argument functions in this framework. . .

If the domain of a function f is a Cartesian product

$$A_1 \times \cdots \times A_n$$

we say f has *arity* n and that f takes n arguments. Moreover, for each

$$(x_1, \dots, x_n) \in A_1 \times \cdots \times A_n,$$

we write

$$f(x_1, \dots, x_n)$$

to denote the value of f at (x_1, \dots, x_n) .

Of course functions that take two arguments are called *binary* functions. And those that take three arguments are called *ternary*.

Range, image and pre-image

Let f be a function from A to B ...

The *range* of f is defined as follows.

$$\text{range}(f) = \{ f(x) \mid x \in A \}$$

Notice that $\text{range}(f) \subset B$.

For any $S \subset A$, the *image* of S (under f) is

$$f(S) = \{ f(x) \mid x \in S \}.$$

Notice that $f(A) = \text{range}(f)$.

Notice also that this notation also allows us to think of f as a function from $\text{power}(A)$ to $\text{power}(B)$. (Don't let this confuse you.)

For any $T \subset B$, the *pre-image* of T (under f) is

$$f^{-1}(T) = \{ x \in A \mid f(x) \in T \}.$$

Notice that $f^{-1}(B) = A$. (Do you see why?)

Notice also that this notation also allows us to think of f^{-1} as a function from $\text{power}(B)$ to $\text{power}(A)$.

Total functions vs partial functions

Recall: A (total) *function* f from A to B is a relation on $A \times B$ such that

for all $x \in A$, there is **exactly** one $y \in B$ s.t. $(x, y) \in f$.

A standard alternative definition allows functions to be “undefined” on some elements of the domain...

A *partial function* f from A to B is a relation on $A \times B$ such that

for all $x \in A$, there is **at most** one $y \in B$ s.t. $(x, y) \in f$.

A familiar example is real division.

We will typically be interested in total functions.

If A and B are nonempty finite sets, how many partial functions of type $A \rightarrow B$?

Equality of functions

For functions to be the same (that is, equal), they must have the same type and they must agree on all values.

So, functions f and g from A to B are equal if

$$\text{for all } x \in A, f(x) = g(x).$$

Consider the following two functions from \mathcal{N} to \mathcal{N} :

$$\sum_{i=0}^n i \qquad \frac{n(n+1)}{2}$$

Are they equal?

How about the following functions of type

$$A^* \times A^* \rightarrow \text{power}(A^*)$$

(where A is an alphabet)?

$$f(x, y) = \{xx, xy, yx, yy\}$$
$$g(x, y) = \{zw \mid (z, w) \in \{x, y\} \times \{x, y\}\}$$

Injective functions

A function $f : A \rightarrow B$ is *injective* if

for all $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

Or, in words: distinct domain elements get distinct values.

Let $f : \mathcal{Z} \rightarrow \mathcal{Z}$ be s.t. $f(x) = 2x$. Is this injective?

Let $f : \mathcal{Z} \rightarrow \mathcal{Z}$ be s.t. $f(x) = x^2$. Is this injective?

How about the following function from \mathcal{Z} to \mathcal{N} ?

$$f(x) = \begin{cases} 2x & , \text{ if } x \geq 0 \\ -2x - 1 & , \text{ otherwise} \end{cases}$$

How about the following function from $\mathcal{N} \times \mathcal{N}$ to \mathcal{N} ?

$$f(x, y) = \frac{(x + y)^2 + 3x + y}{2}$$

(Cantor's pairing function, p. 124)

Surjective functions

A function $f : A \rightarrow B$ is *surjective* if

$$\text{range}(f) = B.$$

That is, if

for every $y \in B$ there is an $x \in A$ s.t. $f(x) = y$.

Or, in words:

every codomain element is the value of some domain element.

Let $f : \mathcal{N} \rightarrow \mathcal{Z}^+$ be the function such that

$$f(x) = x + 1.$$

Is f surjective?

What if we take f instead to be the function of type $\mathcal{N} \rightarrow \mathcal{N}$ that satisfies the above equation (for all natural numbers x)?

How about this function (from the previous page) from \mathcal{Z} to \mathcal{N} ?

$$f(x) = \begin{cases} 2x & , \text{ if } x \geq 0 \\ -2x - 1 & , \text{ otherwise} \end{cases}$$

Claim. If A is a nonempty set, then

there is an injection from A to B

iff

there is a surjection from B to A .

[See textbook for proof, preferably after you've first tried to prove it yourself. (Notice that the text doesn't mention nonemptiness of A in the statement of the theorem, but if you look carefully you will see that nonemptiness is needed, and used, in the proof. (Do you see where? It is easy to miss.))]

Proof idea...

Bijections

A function is *bijective* if it is injective and surjective.

A bijective function is called a *bijection* or a *one-to-one correspondence*.

Observe that $f : A \rightarrow B$ is injective iff

for all $y \in B$ there is **at most one** $x \in A$ s.t. $f(x) = y$.

Similarly, $f : A \rightarrow B$ is surjective iff

for all $y \in B$ there is **at least one** $x \in A$ s.t. $f(x) = y$.

Therefore: a function $f : A \rightarrow B$ is bijective iff

for all $y \in B$ there is **exactly one** $x \in A$ s.t. $f(x) = y$.

The function $f : \mathcal{N} \rightarrow \mathcal{Z}^+$ defined by $f(x) = x + 1$ is a bijection.

The function f from even integers to odd integers defined by $f(x) = x + 1$ is also a bijection.

What if f instead has type $\mathcal{N} \rightarrow \mathcal{N}$? $\mathcal{Z} \rightarrow \mathcal{Z}$?

Consider the conditions that a relation R on $A \times B$ must satisfy in order to be a bijection:

- ▶ R is a function from A to B iff

for all $x \in A$ there is exactly one $y \in B$ s.t. $(x, y) \in R$

- ▶ a function $R : A \rightarrow B$ is bijective iff

for all $y \in B$ there is exactly one $x \in A$ s.t. $(x, y) \in R$

So it is clear why a bijection is also called a “one-to-one correspondence”: it is a relation that establishes a one-to-one correspondence between the elements of sets A and B .

Let $(0, 1) = \{x \in \mathcal{R} \mid 0 < x < 1\}$. (So here $(0, 1)$ denotes a set of reals.)

Claim: The function $f : (0, 1) \rightarrow \mathcal{R}^+$ defined below is bijective.

$$f(x) = \frac{x}{1-x}$$

Proof. To show that f is injective, assume that $f(x) = f(y)$.
(So what is our obligation now?) Then

$$\frac{x}{1-x} = \frac{y}{1-y},$$

so

$$x - xy = y - yx$$

which shows that $x = y$.

To show that f is surjective, take any $y \in \mathcal{R}^+$. (What is our obligation now?)

So we need an $x \in (0, 1)$ s.t. $f(x) = y$. That is, we want

$$\frac{x}{1-x} = y.$$

Solving for x :

$$x = y - xy$$

$$x + xy = y$$

$$x(y+1) = y$$

$$x = \frac{y}{y+1}$$

It remains only to notice that since $y > 0$, $0 < x < 1$.

Inverse of a bijection

Claim: If $f : A \rightarrow B$ is a bijection, then there is a unique function $g : B \rightarrow A$ such that, for all $y \in B$ and $x \in A$,

$$g(y) = x \text{ iff } f(x) = y.$$

This function is called the *inverse* of f , and is denoted by f^{-1} .

The function f^{-1} too is bijective, and its inverse is f .

The definition of inverse of a bijection relies on the claim: if f is a bijection, then such an inverse **exists** and **unique**.

How hard is it to check that this claim holds? ...

First let's check that every bijection has an inverse.

Consider the alternative characterization of a bijection:

A function $f : A \rightarrow B$ is bijective iff

for all $y \in B$ there is exactly one $x \in A$ s.t. $f(x) = y$.

Notice that this is precisely to say that the relation

$$R = \{ (y, x) \mid y \in B, x \in A, f(x) = y \}$$

is a function from B to A : that is, for every $y \in B$ there is exactly one $x \in A$ such that $(y, x) \in R$.

And what about uniqueness of the inverse?

First notice that if g is an inverse of $f : A \rightarrow B$, then for all $x \in A$,

$$g(f(x)) = x$$

and for all $y \in B$

$$f(g(y)) = y.$$

Now, let g and h be inverses of $f : A \rightarrow B$. Then for any $y \in B$ we have

$$\begin{aligned} g(y) &= g(f(h(y))) && (h \text{ is an inverse of } f) \\ &= h(y) && (g \text{ is an inverse of } f) \end{aligned}$$

Function composition

For $g : A \rightarrow B$ and $f : C \rightarrow D$, with $B \subset C$, we define the *composition* of f and g , denoted $f \circ g$, as the function from $A \rightarrow D$ s.t.

$$(f \circ g)(x) = f(g(x)).$$

For example, if f and g are functions of type $\mathcal{N} \rightarrow \mathcal{N}$ s.t.

$$f(x) = x + 1 \quad \text{and} \quad g(x) = 2^x$$

then $f \circ g$ is the function of type $\mathcal{N} \rightarrow \mathcal{N}$ s.t.

$$(f \circ g)(x) = 2^x + 1$$

and $g \circ f$ is the function of type $\mathcal{N} \rightarrow \mathcal{N}$ s.t.

$$(g \circ f)(x) = 2^{x+1}.$$

Notice that for all bijections $f : A \rightarrow B$,

$$(f^{-1} \circ f)(x) = x$$

for all $x \in A$, and

$$(f \circ f^{-1})(y) = y$$

for all $y \in B$.

That is to say,

$$f^{-1} \circ f = \{(x, x) \mid x \in A\}$$

and

$$f \circ f^{-1} = \{(y, y) \mid y \in B\}.$$

Claim. Consider any $f : A \rightarrow B$ and $g : C \rightarrow D$, with $B \subset C$. If f and g are injective, then $g \circ f$ is injective.

Let's prove this. . .

Assume f and g are injective. [What is our obligation now?]

Take any $x, y \in A$ s.t. $x \neq y$. [What is our obligation now?]

Since f is injective, it follows that $f(x) \neq f(y)$.

And since g is also injective, we have $g(f(x)) \neq g(f(y))$.

That is to say, $(g \circ f)(x) \neq (g \circ f)(y)$.

Claim. Consider any $f : A \rightarrow B$ and $g : B \rightarrow C$.

- ▶ If f and g are surjective, then $g \circ f$ is surjective.
- ▶ If f and g are bijective, then $g \circ f$ is bijective.

Can you prove this?

Pigeonhole principle

For finite sets A, B , if $|A| > |B|$, then there is no injective function from A to B .

Clearly this holds if $B = \emptyset$. (Can you explain why?)

And if $|A| > |B| > 0$, then for any function $f : A \rightarrow B$, there are $x, y \in A$ s.t. $x \neq y$ and $f(x) = f(y)$.

Example: Any set of 17 nonempty strings over $\{a, b, c, d\}$ includes two different strings whose first symbols agree and whose last symbols also agree.

Let A be a set of 17 nonempty strings over $\{a, b, c, d\}$.

Let $B = \{a, b, c, d\} \times \{a, b, c, d\}$.

Let f be the function from A to B s.t.

$$f(x) = (s, t) \quad \text{iff} \quad x \text{ begins with } s \text{ and ends with } t.$$

This function is not injective. How do we know?

Notice that $|A| = 17$ and $|B| = 16$. Thus, by the pigeonhole principle, **no** function from A to B is injective.