

Comparing sizes of sets

Sets A and B are the *same size* if there is a bijection from A to B .

(That was a definition!)

For finite sets A, B , it is not difficult to verify that
there is a bijection from A to B iff $|A| = |B|$.

Let's do it. . .

Take arbitrary finite sets A and B .

LR: Assume $f : A \rightarrow B$ is bijective.

Then f is injective.

So, by the pigeonhole principle, $|A| \leq |B|$.

Also $f^{-1} : B \rightarrow A$ is injective. [Do you follow this step?]

So, again by the pigeonhole principle, $|B| \leq |A|$.

We can conclude that $|A| = |B|$.

RL: Assume that $|A| = |B|$. Since A is finite, there is a bijection $f : A \rightarrow \{1, \dots, |A|\}$. And since B is also finite, there is a similar bijection $g : B \rightarrow \{1, \dots, |B|\}$. Moreover, since $|A| = |B|$, the codomains of bijections f and g are the same. It follows that $g^{-1} \circ f$ is a bijection from A to B .

Let's not write $|S|$ when S is an infinite set

The textbook proposes the notation

$$|A| = |B|$$

to say that A and B are the same size. But this is bad notation when A or B is infinite!

Why? Because $|A|$ is not defined for an infinite set A .

What do I mean? Well, it won't do, for instance, to say that $|A| = \infty$ whenever A is infinite. Why? Because then any two infinite sets would be the same size!

But aren't infinite sets all "the same size"?

(Namely, infinite.)

Well no, but that's a long story. . .

Countable sets

A set is *countable* if it is finite or is the same size as \mathcal{N} .

So countable sets can be either finite or infinite.

The obvious question is: Are there any sets that are *not* countable?

Short answer: Yes.

Familiar example of an uncountable set: The set of real numbers.

Or the open interval of real numbers between 0 and 1.

Or $\text{power}(\mathcal{N})$.

Or $\text{power}(\{0, 1\}^*)$.

Soon we will learn a method — called “diagonalization” — for proving that one infinite set is larger than another.

But first let's get a firmer understanding of countability...

Countability as “enumerability”

Pick a natural number k . If you start listing the natural numbers in their “standard” order — that is, enumerating them — you will reach k in a finite number of steps (namely, $k + 1$ steps).

Intuitively, this is an argument that \mathcal{N} is “enumerable”, or countable.

Similarly, take any set S for which there is a bijection

$$f : S \rightarrow \mathcal{N}.$$

Each element x of S corresponds to a natural number $f(x)$.

If you start listing the elements of S in the order given by f ($f^{-1}(0), f^{-1}(1), f^{-1}(2) \dots$), you will reach x (for any given x) in a finite number of steps (namely $f(x) + 1$ steps).

This is the correct understanding of enumerability.

So how about this (**fallacious**) argument? To see that the real numbers are countable, do the following. Take an arbitrary real number x . Now start listing real numbers, one by one, and after some number of steps, include x as the next element in the list. Since you wrote x within a finite number of steps, the set of real numbers is countable.

Every subset of \mathcal{N} is countable

Claim: Every subset of the natural numbers is countable.

Proof: Take any subset S of \mathcal{N} . [So what is our goal now?]

If S is finite, we're done. So assume S is infinite.

Notice that for each $x \in S$, $\{n \mid n \in S, n < x\}$ is finite.

Take $f : S \rightarrow \mathcal{N}$ as follows. For all $x \in S$,

$$f(x) = |\{n \mid n \in S, n < x\}|.$$

To see that f is injective, consider any two distinct elements x, y of S .

Wlog assume that $x < y$. Then $f(x) < f(y)$. To see this, notice that

$$y \notin \{n \mid n \in S, n < x\}$$

while

$$\{n \mid n \in S, n < x\} \subset \{n \mid n \in S, n < y\}.$$

To see that f is surjective, take any $k \in \mathcal{N}$. Let A be the set consisting of the $k + 1$ smallest elements of S . (That is, $|A| = k + 1$ and every element of A is less than every element of $S - A$.) Let x be the largest element of A , and notice that $f(x) = k$.

Proving countability

To show that A is countable, it is sufficient to show that there is an injection from A to \mathcal{N} .

Indeed, if $f : A \rightarrow \mathcal{N}$ is injective, then there is a bijection

$$g : A \rightarrow \text{range}(f)$$

such that, for all $x \in A$,

$$g(x) = f(x).$$

[How hard is it to see that g is bijective?] The existence of such a g shows that A is the same size as $\text{range}(f)$, which is a subset of \mathcal{N} , and so, a countable set.

Or, equivalently, it suffices to show that there is a surjection from \mathcal{N} to A .

Why? (Because this implies that there is an injection from A to \mathcal{N} .)

Every subset of a countable set is countable

Claim: Every subset of a countable set is countable.

Proof: Let A be a subset of countable set S .

Since S is countable, there is an injection $f : S \rightarrow \mathcal{N}$.

Take $g : A \rightarrow \mathcal{N}$ s.t. for all $x \in A$,

$$g(x) = f(x).$$

Then g is an injection from $A \rightarrow \mathcal{N}$, which shows that A is countable

Proving countability (more generally)

To show that A is countable, it is sufficient to show that there is an injection from A to **some** countable set.

To see this, assume that B is countable, and that

$$f : A \rightarrow B$$

is injective. Since B is countable, there is an injection

$$g : B \rightarrow \mathcal{N}.$$

It follows that $(g \circ f)$ is an injection from A to \mathcal{N} , from which we can conclude that A is countable.

And from this it follows that we can also prove A countable by showing that there is a surjection from some countable set to A .

[Why?]

The image of a countable set is countable

Claim: The image of a countable set (under any function) is countable.

Proof: Let f be a function from A to B .

Assume that S is a countable subset of A .

[So what is our goal now?]

Take $g : S \rightarrow f(S)$ s.t. for all $x \in S$,

$$g(x) = f(x).$$

Notice that g is a surjection from a countable set, namely S , to $f(S)$.

From this we can conclude that $f(S)$ is countable.

$\mathcal{N} \times \mathcal{N}$ is countable

Claim: $\mathcal{N} \times \mathcal{N}$ is countable.

Proof idea:

$$\begin{array}{lcl} & (0, 0) & \leftrightarrow 0 \\ & (0, 1), (1, 0) & \leftrightarrow 1, 2 \\ & (0, 2), (1, 1), (2, 0) & \leftrightarrow 3, 4, 5 \\ & (0, 3), (1, 2), (2, 1), (3, 0) & \leftrightarrow 6, 7, 8, 9 \\ (0, 4), (1, 3), (2, 2), (3, 1), (4, 0) & \leftrightarrow & 10, 11, 12, 13, 14 \\ & \vdots & \\ & (0, n), (1, n-1), \dots, (n, 0) & \leftrightarrow \sum_{i=0}^n i, \dots, (\sum_{i=0}^n i) + n \\ & \vdots & \end{array}$$

This bijection is given by Cantor's pairing function, mentioned in the last set of lecture notes as an example of an injective function.

$$\begin{aligned} f(x, y) &= \left(\sum_{i=0}^{x+y} i \right) + x \\ &= \frac{(x+y)(x+y+1)}{2} + x \\ &= \frac{x^2 + xy + x + yx + y^2 + y}{2} + \frac{2x}{2} \\ &= \frac{(x+y)^2 + 3x + y}{2} \end{aligned}$$

Every “countable union” of countable sets is countable

Claim: If S_0, S_1, \dots is a sequence of countable sets, then

$$\bigcup_{n \in \mathcal{N}} S_n$$

is also countable.

Proof: For each set S_i , let f_i be a surjection from \mathcal{N} to S_i . (Such a function f_i exists, since S_i is countable.) Take

$$g : \mathcal{N} \times \mathcal{N} \rightarrow \bigcup_{n \in \mathcal{N}} S_n$$

s.t. for all $m, n \in \mathcal{N}$,

$$g(m, n) = f_m(n).$$

Observe that g is surjective. Indeed, take any

$$x \in \bigcup_{n \in \mathcal{N}} S_n.$$

Then, for some $m \in \mathcal{N}$, $x \in S_m$.

And since $f_m : \mathcal{N} \rightarrow S_m$ is surjective, there is an $n \in \mathcal{N}$ s.t. $f_m(n) = x$.

And since g is a surjection from the countable set $\mathcal{N} \times \mathcal{N}$, we can conclude that $\bigcup_{n \in \mathcal{N}} S_n$ is countable.

The rational numbers are countable

In the last set of lecture notes, we considered a bijection between the integers and the natural numbers. The existence of such a function shows that the integers are countable.

Now, let's show that the rational numbers are countable.

We'll do this by representing them as a countable union of countable sets...

For each positive integer d , let

$$S_d = \left\{ \frac{n}{d} \mid n \in \mathcal{Z} \right\}.$$

Since the integers are countable, so is S_d (for every $d \in \mathcal{Z}^+$).

Let $S_0 = \emptyset$.

Now the set of rational numbers can be written as a countable union of countable sets, as follows.

$$\bigcup_{d \in \mathcal{N}} S_d$$

And this shows that the set of rational numbers is indeed countable.

The set of all strings (over any alphabet) is countable

Recall: An alphabet is a finite set of symbols, and for any alphabet A , A^* is the set of all strings over A .

If A is empty, then $A^* =$

If A is a singleton, then it is still easy to see that A^* is countable.

Indeed, take $f : A^* \rightarrow \mathcal{N}$ s.t. for all $x \in A^*$, $f(x) = |x|$.

If $|A| > 1$, we need a different approach.

As it happens though, the definition of closure is perfect for this:

$$A^* = \bigcup_{n \in \mathcal{N}} A^n$$

Notice that, for all $n \in \mathcal{N}$, A^n is countable (in fact, finite).

Thus, A^* is a countable union of countable sets, and so, countable.

So, is every language (over every alphabet) countable? Why?