Comparing sizes of sets

Sets A and B are the *same size* if there is a bijection from A to B.

(That was a definition!)

For finite sets A, B, it is not difficult to verify that

there is a bijection from A to B iff $|A| = |B|$.

Let's do it...

Take arbitrary finite sets A and B.

LR: Assume $f : A \to B$ is bijective.
Then f is injective.
So, by the pigeonhole principle, $|A| \leq |B|$.
Also $f^{-1} : B \to A$ is injective. [Do you follow this step?]
So, again by the pigeonhole principle, $|B| \leq |A|$.
We can conclude that $|A| = |B|$.

RL: Assume that $|A| = |B|$. Since A is finite, there is a bijection $f : A \to \{1, ..., |A|\}$. And since B is also finite, there is a similar bijection $g : B \to \{1, ..., |B|\}$. Moreover, since $|A| = |B|$, the codomains of bijections f and g are the same. It follows that $g^{-1} \circ f$ is a bijection from A to B.

Let's not write $|S|$ when S is an infinite set

The textbook proposes the notation

$$|A| = |B|$$

to say that A and B are the same size. But this is bad notation when A or B is infinite!

Why? Because $|A|$ is not defined for an infinite set A.

What do I mean? Well, it won't do, for instance, to say that $|A| = \infty$ whenever A is infinite. Why? Because then any two infinite sets would be the same size!

But aren't infinite sets all "the same size"?

(Namely, infinite.)

Well no, but that's a long story...

Countable sets

A set is *countable* if it is finite or is the same size as $\mathcal{N}$.

So countable sets can be either finite or infinite.

The obvious question is: Are there any sets that are *not* countable?

Short answer: Yes.

Familiar example of an uncountable set: The set of real numbers.

Or the open interval of real numbers between 0 and 1.

Or $\mathrm{power}(\mathcal{N})$.

Or $\mathrm{power}(\{0, 1\}^*)$.

Soon we will learn a method — called "diagonalization" — for proving that one infinite set is larger than another.

But first let's get a firmer understanding of countability...

Countability as "enumerability"

Pick a natural number k. If you start listing the natural numbers in their "standard" order — that is, enumerating them — you will reach k in a finite number of steps (namely, $k + 1$ steps).

Intuitively, this is an argument that $\mathcal{N}$ is "enumerable", or countable.

Similarly, take any set S for which there is a bijection

$$f : S \rightarrow \mathcal{N}.$$

Each element x of S corresponds to a natural number $f(x)$.

If you start listing the elements of S in the order given by f $(f^{-1}(0), f^{-1}(1), f^{-1}(2) \ldots)$, you will reach x (for any given x) in a finite number of steps (namely $f(x) + 1$ steps).

This is the correct understanding of enumerability.

So how about this (**fallacious**) argument? To see that the real numbers are countable, do the following. Take an arbitrary real number x. Now start listing real numbers, one by one, and after some number of steps, include x as the next element in the list. Since you wrote x within a finite number of steps, the set of real numbers is countable.

Every subset of $\mathcal{N}$ is countable

Claim: Every subset of the natural numbers is countable.

Proof: Take any subset S of $\mathcal{N}$. [So what is our goal now?]

If S is finite, we're done. So assume S is infinite.

Notice that for each $x \in S$, $\{\, n \mid n \in S, n < x \,\}$ is finite.

Take $f : S \rightarrow \mathcal{N}$ as follows. For all $x \in S$,

$$f(x) = |\, \{\, n \mid n \in S, n < x \,\} \,| \,.$$

To see that f is injective, consider any two distinct elements x, y of S. Wlog assume that $x < y$. Then $f(x) < f(y)$. To see this, notice that

$$y \notin \{\, n \mid n \in S, n < x \,\}$$

while

$$\{\, n \mid n \in S, n < x \,\} \subset \{\, n \mid n \in S, n < y \,\} \,.$$

To see that f is surjective, take any $k \in \mathcal{N}$. Let A be the set consisting of the $k + 1$ smallest elements of S. (That is, $|A| = k + 1$ and every element of A is less than every element of $S - A$.) Let x be the largest element of A, and notice that $f(x) = k$.

Proving countability

To show that A is countable, it is sufficient to show that there is an injection from A to $\mathcal{N}$.

Indeed, if $f : A \to \mathcal{N}$ is injective, then there is a bijection

$$g : A \to \mathrm{range}(f)$$

such that, for all $x \in A$,

$$g(x) = f(x) \,.$$

[How hard is it to see that g is bijective?] The existence of such a g shows that A is the same size as $\mathrm{range}(f)$, which is a subset of $\mathcal{N}$, and so, a countable set.

Or, equivalently, it suffices to show that there is a surjection from $\mathcal{N}$ to A.

Why? (Because this implies that there is an injection from A to $\mathcal{N}$.)

Every subset of a countable set is countable

Claim: Every subset of a countable set is countable.

Proof: Let A be a subset of countable set S.

Since S is countable, there is an injection $f : S \to \mathcal{N}$.

Take $g : A \to \mathcal{N}$ s.t. for all $x \in A$,

$$g(x) = f(x) \, .$$

Then g is an injection from $A \to \mathcal{N}$, which shows that A is countable

Proving countability (more generally)

To show that A is countable, it is sufficient to show that there is an injection from A to **some** countable set.

To see this, assume that B is countable, and that

$$f : A \longrightarrow B$$

is injective. Since B is countable, there is an injection

$$g : B \longrightarrow \mathcal{N} .$$

It follows that $(g \circ f)$ is an injection from A to $\mathcal{N}$, from which we can conclude that A is countable.

And from this it follows that we can also prove A countable by showing that there is a surjection from some countable set to A.

[Why?]

The image of a countable set is countable

Claim: The image of a countable set (under any function) is countable.

Proof: Let f be a function from A to B.

Assume that S is a countable subset of A.

[So what is our goal now?]

Take $g : S \to f(S)$ s.t. for all $x \in S$,

$$g(x) = f(x) .$$

Notice that g is a surjection from a countable set, namely S, to $f(S)$.

From this we can conclude that $f(S)$ is countable.

$\mathcal{N} \times \mathcal{N}$ is countable

Claim: $\mathcal{N} \times \mathcal{N}$ is countable.

Proof idea:

$$
\begin{array}{rcl}
(0,0) & \leftrightarrow & 0 \\
(0,1),(1,0) & \leftrightarrow & 1,2 \\
(0,2),(1,1),(2,0) & \leftrightarrow & 3,4,5 \\
(0,3),(1,2),(2,1),(3,0) & \leftrightarrow & 6,7,8,9 \\
(0,4),(1,3),(2,2),(3,1),(4,0) & \leftrightarrow & 10,11,12,13,14 \\
& \vdots & \\
(0,n),(1,n-1),\ldots,(n,0) & \leftrightarrow & \sum_{i=0}^{n} i,\ldots,\left(\sum_{i=0}^{n} i\right)+n \\
& \vdots &
\end{array}
$$

This bijection is given by Cantor's pairing function, mentioned in the last set of lecture notes as an example of an injective function.

$$
\begin{aligned}
f(x,y) &= \left(\sum_{i=0}^{x+y} i\right) + x \\
&= \frac{(x+y)(x+y+1)}{2} + x \\
&= \frac{x^2+xy+x+yx+y^2+y}{2} + \frac{2x}{2} \\
&= \frac{(x+y)^2+3x+y}{2}
\end{aligned}
$$

Every "countable union" of countable sets is countable

Claim: If $S_0, S_1, \ldots$ is a sequence of countable sets, then

$$\bigcup_{n \in \mathcal{N}} S_n$$

is also countable.

Proof: For each set S_i, let f_i be a surjection from $\mathcal{N}$ to S_i. (Such a function f_i exists, since S_i is countable.) Take

$$g : \mathcal{N} \times \mathcal{N} \to \bigcup_{n \in \mathcal{N}} S_n$$

s.t. for all $m, n \in \mathcal{N}$,

$$g(m, n) = f_m(n) \,.$$

Observe that g is surjective. Indeed, take any

$$x \in \bigcup_{n \in \mathcal{N}} S_n \,.$$

Then, for some $m \in \mathcal{N}$, $x \in S_m$.

And since $f_m : \mathcal{N} \to S_m$ is surjective, there is an $n \in \mathcal{N}$ s.t. $f_m(n) = x$.

And since g is a surjection from the countable set $\mathcal{N} \times \mathcal{N}$, we can conclude that $\bigcup_{n \in \mathcal{N}} S_n$ is countable.

The rational numbers are countable

In the last set of lecture notes, we considered a bijection between the integers and the natural numbers. The existence of such a function shows that the integers are countable.

Now, let's show that the rational numbers are countable.

We'll do this by representing them as a countable union of countable sets...

For each positive integer d, let

$$S_d = \left\{ \frac{n}{d} \mid n \in \mathcal{Z} \right\} .$$

Since the integers are countable, so is S_d (for every $d \in \mathcal{Z}^+$).

Let $S_0 = \emptyset$.

Now the set of rational numbers can be written as a countable union of countable sets, as follows.

$$\bigcup_{d \in \mathcal{N}} S_d$$

And this shows that the set of rational numbers is indeed countable.

The set of all strings (over any alphabet) is countable

Recall: An alphabet is a finite set of symbols, and for any alphabet A, A^* is the set of all strings over A.

If A is empty, then $A^* =$

If A is a singleton, then it is still easy to see that A^* is countable.

Indeed, take $f : A^* \to \mathcal{N}$ s.t. for all $x \in A^*$, $f(x) = |x|$.

If $|A| > 1$, we need a different approach.

As it happens though, the definition of closure is perfect for this:

$$A^* = \bigcup_{n \in \mathcal{N}} A^n$$

Notice that, for all $n \in \mathcal{N}$, A^n is countable (in fact, finite).

Thus, A^* is a countable union of countable sets, and so, countable.

So, is *every* language (over every alphabet) countable? Why?