

Inductively defined sets

Take

$$A = \{3, 5, 7, \dots\}.$$

Although we can't be entirely certain, presumably this means that

$$A = \{2n + 3 \mid n \in \mathcal{N}\}.$$

Another way to describe A is to say:

$$3 \in A \text{ and, for all } n \in \mathcal{N}, \text{ if } n \in A, \text{ then } n + 2 \in A.$$

If we want to understand this last description of A as a definition, it has three (!) parts, as follows:

- ▶ There is an “initial” element of A , namely 3.
- ▶ You construct additional elements of A by adding 2 to any element of A .
- ▶ Nothing else belongs to A .

We call this an “inductive definition” of A .

General form of inductive definition of a set

An inductive definition of a set S has the following form:

- ▶ *Basis*: Specify one or more “initial” elements of S .
- ▶ *Induction*: Give one or more rules for constructing “new” elements of S from “old” elements of S .
- ▶ *Closure*: Understand that S consists of exactly the elements that can be obtained by starting with the initial elements of S and applying the rules for constructing new elements of S .

Typically the closure condition is assumed (that is, left *unstated*), since it is standard.

There is another, more mathematically elegant way to understand the closure condition: **S is the least set satisfying both the basis and induction conditions.**

Another way to understand this: **S is the intersection of all sets that satisfy both the basis and induction conditions.**

Example: Let S be defined as follows:

- ▶ *Basis:* $0 \in S$.
- ▶ *Induction:* If $n \in S$, then $n + 1 \in S$.

Then $S = \mathcal{N}$.

And we can check this. How? We can verify that \mathcal{N} is the *least* set that satisfies the basis and induction conditions in the definition of the set S . [Actually, we'll check that \mathcal{N} is minimal, which for reasons we won't fully explain (yet?), guarantees that it is least.]

1. \mathcal{N} satisfies the basis condition.
2. \mathcal{N} satisfies the induction condition.
3. No proper subset of \mathcal{N} satisfies both conditions. Let's check this...

Take any proper subset X of \mathcal{N} . [We need to show that X doesn't satisfy both conditions. Why?] There is a *least* natural number n missing from X . (That's a powerful claim.) Consider two cases. [Why are these cases exhaustive?]

Case 1: $n = 0$. Then X doesn't satisfy the basis condition.

Case 2: $n = k + 1$ for some $k \in X$. Then X doesn't satisfy the induction condition. (Why?)

Another example: Let S be defined as follows:

- ▶ *Basis:* $0 \in S$.
- ▶ *Induction:* If $n \in S$, then $2n + 1 \in S$.

So what is S ?

Notice: $2^0 - 1 = 0$, and for all $x \in \mathcal{N}$,

$$2^{x+1} - 1 = 2(2^x - 1) + 1.$$

An inductive definition of A^* and other languages

Let A^* be defined as follows:

- ▶ *Basis:* $\Lambda \in A^*$.
- ▶ *Induction:* If $s \in A^*$ and $x \in A$, then $xs \in A^*$.

[What happens if we replace xs above by sx ?]

Let L be the language over $\{0, 1\}$ defined as follows:

- ▶ *Basis:* $\Lambda \in L$.
- ▶ *Induction:* If $s \in L$, then $0s1 \in L$.

Let L be the language over $\{0, 1\}$ defined as follows:

- ▶ *Basis:*
 1. $\Lambda \in L$.
 2. If $x \in \{0, 1\}$, then $x \in L$.
- ▶ *Induction:* If $s \in L$ and $x \in \{0, 1\}$, then $xsx \in L$.

The set of binary trees over A

Define the set B of *binary trees* over alphabet A as follows:

- ▶ *Basis:* $\langle \rangle \in B$.
- ▶ *Induction:* If $L, R \in B$ and $x \in A$, then $\langle L, x, R \rangle \in B$.

Define the set *Twins* over alphabet A as follows:

- ▶ *Basis:* $\langle \rangle \in \text{Twins}$.
- ▶ *Induction:* If $x \in A$ and $T \in \text{Twins}$, then $\langle T, x, T \rangle \in \text{Twins}$.

For any nonempty binary tree $T = \langle L, x, R \rangle$, let

$$\text{left}(T) = L, \quad \text{root}(T) = x, \quad \text{right}(T) = R.$$

Define the set *Opps* over alphabet $\{0, 1\}$ as follows:

- ▶ *Basis*: If $x \in \{0, 1\}$, then $\langle \langle \rangle, x, \langle \rangle \rangle \in \text{Opps}$.
- ▶ *Induction*: If $x, y \in \{0, 1\}$, $T \in \text{Opps}$, and $y \neq \text{root}(T)$, then $\langle T, x, \langle \text{right}(T), y, \text{left}(T) \rangle \rangle \in \text{Opps}$.

Define the set *F* of subsets of \mathcal{N} as follows:

- ▶ *Basis*: $\emptyset \in F$.
- ▶ *Induction*: If $n \in \mathcal{N}$ and $S \in F$, then $S \cup \{n\} \in F$.

What is *F*?

Can we prove this? (By showing that it is a minimal set satisfying the basis and induction conditions.)