

## Recommended problems 5 — CS3512 — Fall '11

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From the textbook exercises for Section 2.1, you should be able to do 1,2,9,15,16,27,28. From Section 2.3, you should be able to do 1–3,4(a,b,e),5(c,f,g),15–18.

As usual, there are also questions raised in lecture notes (and left for you to answer); you should be able to answer these questions, and questions like them.

### Additional problems

1 How many of the relations on  $\{0, 1, 2\} \times \{0, 1, 2, 3\}$  are not functions?

2 Prove that for any function  $f : A \rightarrow B$ , and any  $C, D \subset A$ ,

$$f(C \cap D) \subset f(C) \cap f(D).$$

3 Prove: For any function  $f : A \rightarrow B$  and any  $S \subset A$ ,

$$S \subset f^{-1}(f(S)).$$

Also, give an example of a function  $f$  from  $\{0, 1\}$  to  $\{0, 1\}$  such that

$$\{0\} \neq f^{-1}(f(\{0\}))$$

and

$$\{1\} \neq f^{-1}(f(\{1\})).$$

4 How many injective functions from  $\{0, 1, 2\}$  to  $\{0, 1, 2, 3, 4\}$ ?

5 Let  $f$  be a function from  $A$  to  $B$ , and  $g$  a function from  $B$  to  $C$ .

Prove: If  $f$  and  $g$  are surjective, then  $g \circ f$  is too.

6 Give a counterexample to the following (slightly stronger) claim:

For every function  $f$  from  $A$  to  $B$ , and every function  $g$  from  $C$  to  $D$  s.t.  $B \subset C$ , if  $f$  and  $g$  are surjective, then  $g \circ f$  is too.

7 Give an example of functions  $f : A \rightarrow A$  and  $g : A \rightarrow A$  s.t.

- $f$  is injective, and
- $g$  is surjective, and
- $f \circ g$  is neither injective nor surjective, and
- $g \circ f$  is a bijection.

## Selected example solutions

2 Prove that for any function  $f : A \rightarrow B$ , and any  $C, D \subset A$ ,

$$f(C \cap D) \subset f(C) \cap f(D).$$

Take any  $f : A \rightarrow B$  and  $C, D \subset A$ . [NTS:  $f(C \cap D) \subset f(C) \cap f(D)$ ]

Assume that  $y \in f(C \cap D)$ . [NTS:  $y \in f(C) \cap f(D)$ ]

Since  $y \in f(C \cap D)$ , there is an  $x \in C \cap D$  s.t.  $f(x) = y$ . (Do you follow?)

Notice that  $x \in C$  and  $x \in D$ .

Consequently,  $f(x) = y \in f(C)$  and  $f(x) = y \in f(D)$ . (Do you see why?)

That is,  $y \in f(C) \cap f(D)$ .

3 Prove: For any function  $f : A \rightarrow B$  and any  $S \subset A$ ,

$$S \subset f^{-1}(f(S)).$$

Also, give an example of a function  $f$  from  $\{0, 1\}$  to  $\{0, 1\}$  such that

$$\{0\} \neq f^{-1}(f(\{0\}))$$

and

$$\{1\} \neq f^{-1}(f(\{1\})).$$

Take any  $f : A \rightarrow B$  and any  $S \subset A$ . [NTS:  $S \subset f^{-1}(f(S))$ ]

Assume  $x \in S$ . [NTS:  $x \in f^{-1}(f(S))$ ]

It follows that  $f(x) \in f(S)$ . (Do you see why?)

Since  $x \in S$  and  $S \subset A$ ,  $x \in A$ .

Since  $x \in A$  and  $f(x) \in f(S)$ , we know that  $x \in f^{-1}(f(S))$ . (Do you see why?!)

I'll leave it to you to find the requested example. (How many functions are there from  $\{0, 1\}$  to  $\{0, 1\}$ ? Not many, right? And half of them have the requested properties!)

Notice that such an example shows that the subset inclusion in problem 3 cannot be strengthened to equality. Surprising?

7 Give an example of functions  $f : A \rightarrow A$  and  $g : A \rightarrow A$  s.t.

- $f$  is injective, and
- $g$  is surjective, and
- $f \circ g$  is neither injective nor surjective, and
- $g \circ f$  is a bijection.

Subtle problem... You may suspect that if  $A$  is finite, then every injection  $f$  from  $A$  to  $A$  is bijective, as is every surjection  $g$  from  $A$  to  $A$ . So we need an infinite  $A$ .

Here's one that works. Consider functions  $f$  and  $g$  from  $\mathcal{N}$  to  $\mathcal{N}$  s.t.  
 $f(n) = n + 1$  and  $g(0) = 0$  and  $g(n + 1) = n$ .