

Recommended problems 6 — CS3512 — Fall 09

From the textbook exercises for Section 2.4, you should be able to do 1–3.

Additional problems

- 1 Show that if sets A and B are the same size, then $\text{power}(A)$ and $\text{power}(B)$ are too.
- 2 True or false? For all nonempty sets A, B, C , the set of functions of type $(A \times B) \rightarrow C$ is the same size as the set of functions of type $A \rightarrow (B \rightarrow C)$. Prove your answer correct.
- 3 Show that the set of functions of type $\{0, 1\} \rightarrow \mathcal{N}$ is countable.
- 4 Show that the set of functions of type $\mathcal{N} \rightarrow \{0, 1\}$ is the same size as $\text{power}(\mathcal{N})$.
- 5 Let $L = \{000, 0000\}^+$ and let P be the set of prime numbers. Prove that there is a bijection from L to P . (Hints: Recall that there are infinitely many prime numbers. The argument I have in mind uses: the definition of countable sets, as well as the fact that any subset of a countable set is countable. It also uses: the definition of “same size” for sets, and properties of bijections and function composition.)
- 6 Show that, for any natural number k , the set L of all lists over \mathcal{N} of length k is countable. (Hint: Express L as a countable union of finite sets.)
- 7 Show that $\text{lists}(\mathcal{N})$ is countable. (Hint: Use the result from the previous problem to express $\text{lists}(\mathcal{N})$ as a countable union of countable sets.)

Selected solutions

- 1 Show that if sets A and B are the same size, then $\text{power}(A)$ and $\text{power}(B)$ are too.

The argument here is somewhat involved, although the result should seem unsurprising. Certainly it is clearly true in the finite case. The general case takes some work...

Assume that A and B are sets of the same size. That is, there is a bijection $f : A \rightarrow B$. It will suffice to show that there is a bijection from $\text{power}(A)$ to $\text{power}(B)$. Take $g : \text{power}(A) \rightarrow \text{power}(B)$ s.t. for all $S \in \text{power}(A)$,

$$g(S) = f(S).$$

That is, $g(S)$ is simply the image of S under f . It remains to check that g is bijective. [We do this in two parts, as usual. You might anticipate that will need the fact that f is injective when showing that g is, and similarly, when we argue that g is surjective, we can expect to need that fact that f is.]

To see that g is injective, take any $S, T \in \text{power}(A)$ s.t. $S \neq T$. [NTS: $g(S) \neq g(T)$] Wlog assume there is an $x \in S$ s.t. $x \notin T$. Then $f(x) \in f(S)$. On the other hand, we can show that $f(x) \notin f(T)$. (Indeed, if $f(x)$ belonged to $f(T)$ while $x \notin T$, we could conclude that there is a $y \in T$ s.t. $f(y) = f(x)$, while $y \neq x$, which would contradict the fact that f is injective.) Summing up, we have shown that $f(x) \in f(S)$ and $f(x) \notin f(T)$, from which it follows that $f(S) \neq f(T)$ and so that $g(S) \neq g(T)$.

To see that g is surjective, take any $T \in \text{power}(B)$. [NTS: There is an $S \in \text{power}(A)$ s.t. $g(S) = T$.] Take $S = f^{-1}(T)$. That is, let S be the pre-image of T under f . So $g(S) = f(f^{-1}(T))$. We can complete our argument by showing that $f(f^{-1}(T)) = T$. We have already seen that $f(f^{-1}(T)) \subset T$, so it remains only to show that $T \subset f(f^{-1}(T))$. Take any $y \in T$. [NTS: $y \in f(f^{-1}(T))$] Since f is surjective, there is an $x \in A$ s.t. $f(x) = y$. Notice that $x \in f^{-1}(T)$. It follows that $f(x) \in f(f^{-1}(T))$. That is, $y \in f(f^{-1}(T))$.

2 True or false? For all nonempty sets A, B, C , the set of functions of type $(A \times B) \rightarrow C$ is the same size as the set of functions of type $A \rightarrow (B \rightarrow C)$. Prove your answer correct.

True. This is easily checked for finite A, B, C . For the general case, there is work to do...

Let $g : (A \rightarrow (B \rightarrow C)) \rightarrow ((A \times B) \rightarrow C)$ be s.t. for all functions $f : A \rightarrow (B \rightarrow C)$, the function $g(f) : (A \times B) \rightarrow C$ is s.t. for all $(x, y) \in A \times B$,

$$g(f)(x, y) = f(x)(y).$$

It remains to show that g is bijective.

Take any distinct functions f_1, f_2 of type $A \rightarrow (B \rightarrow C)$. We need to show that $g(f_1) \neq g(f_2)$. Since $f_1 \neq f_2$, there is an $x \in A$ s.t. $f_1(x) \neq f_2(x)$. And since $f_1(x) \neq f_2(x)$, there is a $y \in B$ s.t. $f_1(x)(y) \neq f_2(x)(y)$. Thus we have

$$g(f_1)(x, y) = f_1(x)(y) \neq f_2(x)(y) = g(f_2)(x, y).$$

And from this we can conclude that $g(f_1) \neq g(f_2)$.

Now take any function $h : (A \times B) \rightarrow C$. We need to show that there is a function $f : A \rightarrow (B \rightarrow C)$ s.t. $g(f) = h$. Take f s.t. for all $x \in A$, the function $f(x) : B \rightarrow C$ is s.t. for all $y \in B$

$$f(x)(y) = h(x, y).$$

Then $g(f) = h$, since for all $(x, y) \in A \times B$

$$g(f)(x, y) = f(x)(y) = h(x, y).$$

3 Show that the set of functions of type $\{0, 1\} \rightarrow \mathcal{N}$ is countable.

One way to do this is to prove that there is a surjection from some countable set to the set of functions of type $\{0, 1\} \rightarrow \mathcal{N}$. It is convenient to use the countable set $\mathcal{N} \times \mathcal{N}$ for this purpose.

Accordingly, we specify a surjection

$$f : (\mathcal{N} \times \mathcal{N}) \rightarrow (\{0, 1\} \rightarrow \mathcal{N}).$$

as follows. For all $m, n \in \mathcal{N}$, let

$$f(m, n) = \{(0, m), (1, n)\}.$$

There are two things to notice...

For all $m, n \in \mathcal{N}$, the relation $f(m, n)$ on $\{0, 1\} \times \mathcal{N}$ is indeed a function.

And function f is indeed surjective. To see this, take any function g of type $\{0, 1\} \rightarrow \mathcal{N}$. This function g can be written $\{(0, m), (1, n)\}$ for some $m, n \in \mathcal{N}$, and it is exactly the function $f(m, n) : \{0, 1\} \rightarrow \mathcal{N}$.

Although we do not need it for this problem, it is also easy to see that function f is injective. Indeed, for all $m_1, n_1, m_2, n_2 \in \mathcal{N}$, if $(m_1, n_1) \neq (m_2, n_2)$, then

$$\{(0, m_1), (1, n_1)\} \neq \{(0, m_2), (1, n_2)\}.$$

4 Show that the set of functions of type $\mathcal{N} \rightarrow \{0, 1\}$ is the same size as $\text{power}(\mathcal{N})$.

We need to show that there is a bijection from the set of functions of type $\mathcal{N} \rightarrow \{0, 1\}$ to $\text{power}(\mathcal{N})$. So take

$$f : (\mathcal{N} \rightarrow \{0, 1\}) \rightarrow \text{power}(\mathcal{N})$$

s.t. for all functions $g : \mathcal{N} \rightarrow \{0, 1\}$,

$$f(g) = \{n \mid n \in \mathcal{N}, g(n) = 1\}.$$

Now we should check that f is indeed bijective.

To see that f is injective, take functions g, h of type $\mathcal{N} \rightarrow \{0, 1\}$ s.t. $g \neq h$. [We need to show that $f(g) \neq f(h)$.] Since $g \neq h$, there is some $n \in \mathcal{N}$ for which $g(n) \neq h(n)$. It follows that $n \in f(g)$ iff $n \notin f(h)$. Hence, $f(g) \neq f(h)$.

To see that f is surjective, take an arbitrary $X \in \text{power}(\mathcal{N})$. [We need to show that $X \in \text{range}(f)$.] Let $g : \mathcal{N} \rightarrow \{0, 1\}$ be s.t. for all $n \in \mathcal{N}$,

$$g(n) = 1 \text{ iff } n \in X.$$

Notice that $f(g) = X$.

4 (Here's an alternative solution to 4.) Show that the following two sets are the same size.

$$S = \text{power}(\mathcal{N})$$

$$F = \{f \mid f \text{ is a function from } \mathcal{N} \text{ to } \{0, 1\}\}$$

That is, prove that there is a bijection from S to F . (Specify a bijection from S to F , and *prove* that it is bijective.)

Take $g : S \rightarrow F$ s.t. for all $X \in S$, $g(X)$ is the function from \mathcal{N} to $\{0, 1\}$ s.t. for all $n \in \mathcal{N}$,

$$g(X)(n) = \begin{cases} 1 & , \text{ if } n \in X \\ 0 & , \text{ otherwise} \end{cases}$$

To see that g is injective, take elements $X, Y \in S$ s.t. $X \neq Y$. [We need to show that $g(X) \neq g(Y)$.] Notice first that $g(X)$ and $g(Y)$ are functions of the same type (namely, $\mathcal{N} \rightarrow \{0, 1\}$), so to show that they are not the same function, we need to find some $n \in \mathcal{N}$ s.t. $g(X)(n) \neq g(Y)(n)$. Since $X \neq Y$, there is some natural number n that belongs to exactly one of X, Y . Wlog, assume that $n \in X$ and $n \notin Y$. Then we see that $g(X)(n) = 1$, while $g(Y)(n) = 0$.

To see that g is surjective, take any $f \in F$. [We need to show that there is some $X \in \text{power}(\mathcal{N})$ s.t. $g(X) = f$.] Take

$$X = \{ n \mid n \in \mathcal{N}, f(n) = 1 \}.$$

Of course f and $g(X)$ have the same type, so it remains only to show that, for all $n \in \mathcal{N}$, $g(X)(n) = f(n)$. So take any $n \in \mathcal{N}$. Consider two cases. Case 1: $n \in X$. Then $g(X)(n) = 1 = f(n)$. Case 2: $n \notin X$. Then $g(X)(n) = 0 = f(n)$.

6 Show that, for any natural number k , the set L of all lists over \mathcal{N} of length k is countable. (Hint: Express L as a countable union of finite sets.)

Take any $k \in \mathcal{N}$. For all $n \in \mathcal{N}$, let S_n be the set of all lists over $\{0, \dots, n\}$ whose length is k . Notice that, for each $n \in \mathcal{N}$, $|S_n| = (n+1)^k$. Notice also that every list over \mathcal{N} of length k belongs to S_n for some $n \in \mathcal{N}$. (Indeed, every list over \mathcal{N} of length k belongs to S_n for infinitely many numbers $n \in \mathcal{N}$!) On the other hand, for every $n \in \mathcal{N}$, $S_n \subset L$. We can conclude that

$$L = \bigcup_{n \in \mathcal{N}} S_n.$$

So L is a countable union of countable sets, and so is countable.

7 Show that $\text{lists}(\mathcal{N})$ is countable. (Hint: Use the result from the previous problem to express $\text{lists}(\mathcal{N})$ as a countable union of countable sets.)

For all $n \in \mathcal{N}$, let L_n be the set of all lists over \mathcal{N} of length n . We showed in the previous problem that each of these sets is countable. Notice that every element x of $\text{lists}(\mathcal{N})$ belongs to $L_{|x|}$, and that, for every $n \in \mathcal{N}$, $L_n \subset \text{lists}(\mathcal{N})$. Thus,

$$\text{lists}(\mathcal{N}) = \bigcup_{n \in \mathcal{N}} L_n,$$

which is a countable union of countable sets, and so is countable.