Recommended problems 9 — CS3512 — Fall '11

Our topic this time is mathematical induction.

"Weak" mathematical induction is a special form of structural induction. In the lecture notes, I specify a format for weak induction proofs and provide quite a few examples.

We also studied "strong" mathematical induction. Again, in the lecture notes I specify a format for such proofs and give an example.

In Section 4.4 and the associated exercises, you can find many claims provable by one or the other form of mathematical induction. Look them over, and try some. Make sure to understand and follow the recommended format.

Additional problems, with solutions

1 Recall the recursively-defined sequence of Fibonacci numbers.

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_{n+2} &= F_n + F_{n+1} \end{aligned}$$

Here is a similar recursively-defined sequence, the so-called Lucas numbers.

$$\begin{aligned} L_0 &= 2 \\ L_1 &= 1 \\ L_{n+2} &= L_n + L_{n+1} \end{aligned}$$

Prove the following claim **by strong mathematical induction**.

Claim: For all $n \in \mathcal{N}$, $F_n = L_{n+1} - F_{n+2}$.

Proof by strong mathematical induction on n $(n \geq 0)$.

IH: For all $m \in \mathcal{N}$ s.t. $0 \leq m < n$, $F_m = L_{m+1} - F_{m+2}$.

NTS: $F_n = L_{n+1} - F_{n+2}$.

Consider three cases.

Case 1: $n = 0$. $F_0 = 0 = 1 - 1 = L_1 - F_2$.

Case 2: $n = 1$. $F_1 = 1 = 3 - 2 = L_2 - F_3$.

Case 3: $n \geq 2$.

$$\begin{aligned}
F_n &= F_{n-2} + F_{n-1} && (\text{defn } F_n,\ n \geq 2) \\
&= (L_{n-1} - F_n) + (L_n - F_{n+1}) && (\text{IH, twice}) \\
&= (L_{n-1} + L_n) - (F_n + F_{n+1}) && \\
&= L_{n+1} - F_{n+2} && (\text{defn } L_{n+1},\ \text{defn } F_{n+2},\ n \geq 1)
\end{aligned}$$

2 Assume that for all $n \in \mathcal{Z}$, **if** $P(m)$ for all $m < n$, **then** $P(n)$. Can we conclude that for all $n \in \mathcal{Z}$, $P(n)$? Prove your answer correct.

No, the conclusion does not follow. For example, let $P(n)$ stand for the claim that n is odd. Take an arbitrary integer n. It is not the case that all integers less than n are odd. So the assumption holds for this choice of $P(n)$, but the conclusion does not.

3 Assume that for all $n \in \mathcal{Z}$, **if** $P(m)$ for all $m < n$, **then** $P(n)$. Also assume that for some $n \in \mathcal{Z}$, $P(m)$ for all $m < n$. Can we conclude that for all $n \in \mathcal{Z}$, $P(n)$? Prove your answer correct.

Yes, the conclusion follows. Indeed, from the second assumption, we know that for some $n_0 \in \mathcal{Z}$ we have $P(m)$ for all $m < n_0$. So it will suffice to prove $P(n)$ for all $n \geq n_0$, which is easily done, by strong induction on n, given the first assumption. [You might want to try this.]