**Systems**

- Broadly speaking, a system is anything that responds when stimulated or excited.

- The systems most commonly analyzed by engineers are artificial systems designed by humans.

- Engineering system analysis is the application of mathematical methods to the design and analysis of systems.
Systems

- Systems have inputs and outputs
- Systems accept excitation signals at their inputs and produce response signals at their outputs
- Systems are often usefully represented by block diagrams

A single-input, single-output system block diagram

\[ x(t) \xrightarrow{\mathcal{H}} y(t) \]
Some Examples of Systems

- Automobile Chassis
  - Spring
  - Shock Absorber

- Rotation
  - Wind

\[ x(t) \quad y(t) \]
A Multiple-Input, Multiple-Output System Block Diagram

\[ x_1(t) \xrightarrow{\mathcal{H}_1} y_1(t) \]

\[ x_2(t) \xrightarrow{\mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4} y_2(t) \]
Continuous and Discrete Time Systems

**Continuous Time Systems**

\[ x(t) \rightarrow H \rightarrow y(t) \]

*Example: an RC circuit*

**Discrete Time Systems**

\[ x[n] \rightarrow H \rightarrow y[n] \]

*Example: a delayed adder*
An Electrical Circuit Viewed as a System

1. An RC lowpass filter is a simple electrical system

2. It is excited by a voltage, \( v_{in}(t) \), and responds with a voltage, \( v_{out}(t) \)

3. It can be viewed or modeled as a single-input, single-output system
Response of an RC Lowpass Filter to a Step Excitation

If an RC lowpass filter is excited by a step of voltage,

\[ v_{in}(t) = A u(t) \]

its response is

\[ v_{out}(t) = A \left( 1 - e^{-\frac{t}{RC}} \right) u(t) \]

If the excitation is doubled, the response doubles.
A DT System

If the excitation, $x[n]$, is the unit sequence, the response is

$$y[n] = \left[ 5 - 4 \left( \frac{4}{5} \right)^n \right] u[n]$$

If the excitation is doubled, the response doubles.
Characteristics of a System

1. Homogeneity
2. Additivity
3. Time Invariance
4. Stability
5. Causality

Linear Time-Invariant Systems (LTI Systems)
Homogeneity

Continuous Time Homogeneous

Discrete Time Homogeneous

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Additivity

\[ x_1(t) \rightarrow H \rightarrow y_1(t) \]

\[ x_2(t) \rightarrow H \rightarrow y_2(t) \]

\[ y(t) = y_1(t) + y_2(t) \]
**Linearity**

\[ x_1(t) \rightarrow H \rightarrow y_1(t) \]

\[ x_2(t) \rightarrow H \rightarrow y_2(t) \]

\[ y(t) = \alpha y_1(t) + \beta y_2(t) \]
**Time-Invariance**

\[ x(t) \xrightarrow{H} y(t) \]

\[ x(t) \xrightarrow{\text{Delay} \ t_0} x(t-t_0) \xrightarrow{H} y(t-t_0) \]

**Discrete Time Time-Invariant**

\[ x[n] \xrightarrow{H} y[n] \]

\[ x[n] \xrightarrow{\text{Delay} \ n_0} x[n-n_0] \xrightarrow{H} y[n-n_0] \]
Stability

\[ x(t) \rightarrow H \rightarrow y(t) \]

**Stable Input** → **Stable Output**

**Stable Input means:**

\[ |x(t)| < \infty \quad -\infty < t < \infty \]

**Stable Output means:**

\[ |y(t)| < \infty \quad -\infty < t < \infty \]

also called BIBO Stable
Causality

Output follows input and can not precede input.
Let’s look at Examples of LTI Systems

**Continuous Time**

\[ RC \frac{dy(t)}{dt} + y(t) = x(t) \]

**Discrete Time**

\[ y[n] - \frac{1}{2} y[n-1] = x[n] \]
Idea of Unit Impulse Response

\[ x(t) \rightarrow H \rightarrow y(t) \]

**Continuous Time System**

\[ x(t) = \delta(t) \rightarrow H \rightarrow y(t) = h(t) \]

**Discrete Time System**

\[ x[n] \rightarrow H \rightarrow y[n] \]

\[ x[n] = \delta[n] \rightarrow H \rightarrow y[n] = h[n] \]
Higher Order Discrete System

\[ a_n y[n] + a_{n-1} y[n-1] + \ldots + a_{n-D} y[n-D] = x[n] \]

\[ x[n] = \delta[n] \]

\[ \Rightarrow y[n] = h[n] \]
**Impulse Response to System Response**

\[ a_n y[n] + a_{n-1} y[n-1] + \ldots + a_{n-D} y[n-D] = x[n] \]

\[ x[n] = \delta[n] \quad \Rightarrow \quad y[n] = h[n] \]

*Any Input \( x[n] \) can be written as*

\[ x[n] = \ldots + x[-2] \delta[n+2] + x[-1] \delta[n+1] + \]

\[ x[0] \delta[n] + x[1] \delta[n-1] + x[2] \delta[n-2] + \ldots \]

*This means system response, \( y[n] \) can be given by*

\[ y[n] = \ldots + x[-2] h[n+2] + x[-1] h[n+1] + \]

Simple System Response Example

System Excitation

System Impulse Response

System Response

\[ y[n] = h[n] + h[n - 1] \]
More Complicated System Response Example
**Convolution Sum**


\[ y[n] = \sum_{m=-2}^{m=2} x[m]h[n - m] \]

**Convolution Sum**

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - m] \]

\[ y[n] = x[n] \ast h[n] \]

**Superposition of delayed and weighted impulse responses**
A Convolution Sum Example

\[ x[n] \]

\[ h[n] \]

\[ h[-m] \]

\[ h[n - m] \]
A Convolution Sum Example

\[ n = -1 \]

\[ x[m] \times h[-1 - m] \]

\[ y[-1] = 2 \]

\[ n = 0 \]

\[ x[m] \times h[0 - m] \]

\[ y[0] = 6 \]
A Convolution Sum Example

\[ n = 1 \]

\[ x[m] \quad h[1 - m] \]

\[ y[1] = 6 \]

\[ n = 2 \]

\[ x[m] \quad h[2 - m] \]

\[ y[2] = 4 \]
A Convolution Sum Example

\[ x[n] \]
\[ h[n] \]
\[ y[n] \]
Convolution Integral in Continuous Time

\[ x(t) = \delta(t) \quad \xrightarrow{H} \quad y(t) = h(t) \]

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \]

\[ y(t) = x(t) * h(t) \]

Superposition of delayed and weighted impulse responses
A Graphical Illustration of the Convolution Integral

The convolution integral is defined by

\[ x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \]

For illustration purposes let the excitation, \( x(t) \), and the impulse response, \( h(t) \), be the two functions below.
A Graphical Illustration of the Convolution Integral

In the convolution integral there is a factor, \( h(t - \tau) \)

We can begin to visualize this quantity in the graphs below.
A Graphical Illustration of the Convolution Integral

The functional transformation in going from \( h(\tau) \) to \( h(t - \tau) \) is

\[
h(\tau) \xrightarrow{\tau \rightarrow -\tau} h(-\tau) \xrightarrow{\tau \rightarrow \tau - t} h(-(\tau - t)) = h(t - \tau)
\]
A Graphical Illustration of the Convolution Integral

The convolution value is the area under the product of \( x(t) \) and \( h(t - \tau) \). This area depends on what \( t \) is. First, as an example, let \( t = 5 \).

For this choice of \( t \) the area under the product is zero. If

\[
y(t) = x(t) \ast h(t)
\]

then \( y(5) = 0 \).
A Graphical Illustration of the Convolution Integral

Now let $t = 0$.

Therefore $y(0) = 2$, the area under the product.
A Graphical Illustration of the Convolution Integral

The process of convolving to find $y(t)$ is illustrated below.
Properties of Convolution

**Continuous Time**

\[ h(t) = \int_{-\infty}^{\infty} \delta(\tau) h(t - \tau) d\tau \]

\[ = \delta(t) \ast h(t) \]

**Discrete Time**

\[ h[n] = \sum_{m=-\infty}^{\infty} \delta[m] h[n - m] \]

\[ = \delta[n] \ast h[n] \]
Properties of Convolution ... cont.

**Continuous Time**

\[
y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau
\]

\[
= x(t) * h(t)
\]

\[
= h(t) * x(t)
\]

\[
= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau
\]

**Discrete Time**

\[
y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]
\]

\[
= x[n] * h[n]
\]

\[
= h[n] * x[n]
\]

\[
= \sum_{m=-\infty}^{\infty} h[m]x[n-m]
\]
### Causality and Stability from Impulse Response

#### Continuous Time

**Causality means for** $t < 0$

$$h(t) = 0$$

**Stability means**

$$\int_{-\infty}^{\infty} h(t) dt < \infty$$

**Example:**

$$h(t) = e^{-t/RC} u(t)$$

#### Discrete Time

**Causality means for** $n < 0$

$$h[n] = 0$$

**Stability means**

$$\sum_{n=-\infty}^{\infty} h[n] < \infty$$

**Example:**

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$
Cascaded and Parallel Systems

\[ x(t) \rightarrow h_1(t) \rightarrow y(t) = x(t) * h_1(t) \]

\[ x(t) \rightarrow h_2(t) \rightarrow y(t) = x(t) * h_2(t) \]

Cascaded Systems

\[ x(t) \rightarrow h_1(t) \rightarrow h_2(t) \rightarrow y(t) = x(t) * h_1(t) * h_2(t) \]

Parallel Systems

\[ x(t) \rightarrow h_1(t) \rightarrow h_2(t) \rightarrow y(t) = x(t) *[h_1(t) + h_2(t)] \]
Responses to Standard Signals

Figure 3.70
Relations between integrals and derivatives of excitations and responses for an LTI system.
Finding Impulse Response

**Continuous Time**

\[
+ \quad \begin{array}{c}
\text{R} \\
\text{C}
\end{array} \quad + \\
\begin{array}{c}
x(t) \\
i(t)
\end{array} \quad \begin{array}{c}
y(t)
\end{array} \\
- \\
- \\
\int_{t}^{t} dt RC dy(t) + y(t) = x(t) \\
\Rightarrow RC \frac{dh(t)}{dt} + h(t) = \delta(t) \\
\Rightarrow h(t) = e^{-t/RC} u(t)
\]

**Discrete Time**

\[
\begin{array}{c}
x[n]
\end{array} \quad \rightarrow \\
\begin{array}{c}
D=1
\end{array} \quad \rightarrow \\
\begin{array}{c}
y[n]
\end{array}
\]

\[
y[n] - \frac{1}{2} y[n-1] = x[n] \\
\Rightarrow h[n] - \frac{1}{2} h[n-1] = \delta[n] \\
\Rightarrow h[n] = \left(\frac{1}{2}\right)^{n} u[n]
\]
**Finding the Impulse Response by Recursive Method**

\[ y[n] - \frac{1}{2} y[n-1] = x[n] \]

\[ \Rightarrow y[n] = x[n] + \frac{1}{2} y[n-1] \]

<table>
<thead>
<tr>
<th>n</th>
<th>Unit Impulse</th>
<th>y(n)</th>
<th>h(n)</th>
</tr>
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<tbody>
<tr>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1/64</td>
<td>1/64</td>
</tr>
</tbody>
</table>
Solving First Order Difference Equation

\[ x[n] \xrightarrow{} \bigcirc \xrightarrow{} y[n] \]
\[ x[n] \xrightarrow{\times \frac{1}{2}} y[n] \]\n\[ D=1 \]

\[ y[n] - \frac{1}{2} y[n-1] = x[n] \]

\textbf{Homogeneous Solution}

\[ y[n] - \frac{1}{2} y[n-1] = 0 \]
\[ y[n] = \frac{1}{2} y[n-1] \]
\[ \frac{y[n]}{y[n-1]} = \frac{1}{2} \]
\[ \Rightarrow y[n] = K \left( \frac{1}{2} \right)^n \]
\[ \Rightarrow y[n] = \left( \frac{1}{2} \right)^n u[n] \]

\textbf{Particular Solution}

\[ y[n] - \frac{1}{2} y[n-1] = \delta[n] \]
\[ \text{At } n = 0 \]
\[ y[0] - \frac{1}{2} y[-1] = \delta[0] \]
\[ y[0] - 0 = 1 \]
\[ y[0] = 1 \]
\[ K \left( \frac{1}{2} \right)^0 = 1 \]
\[ \Rightarrow K = 1 \]
Solving First Order Differential Equation

**Homogeneous Solution**

\[ RC \frac{dy(t)}{dt} + y(t) = 0 \]

\[ \frac{dy(t)}{dt} = -\frac{1}{RC} y(t) \]

\[ \Rightarrow y(t) = Ke^{-\frac{t}{RC}} \]

**Particular Solution**

\[ RC \frac{dy(t)}{dt} + y(t) = \delta(t) \]

*Integrating from t = 0⁻ to t = 0⁺*

\[ RC \int_{0^-}^{0^+} \frac{dy(t)}{dt} dt + \int_{0^-}^{0^+} y(t) dt = \int_{0^-}^{0^+} \delta(t) dt \]

\[ RC[y(0^+) - y(0^-)] + \int_{0^-}^{0^+} y(t) dt = 1 \]

\[ RC[y(0^+) - y(0^-)] + 0 = 1 \]
Solving First Order Differential Equation

Homogeneous Solution

\[ RC \frac{dy(t)}{dt} + y(t) = 0 \]

\[ \frac{dy(t)}{dt} = -\frac{1}{RC} y(t) \]

\[ \Rightarrow y(t) = Ke^{-\frac{t}{RC}} \]

\[ \Rightarrow y(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t) \]

Particular Solution … cont

\[ RC[y(0^+) - y(0^-)] + 0 = 1 \]

\[ RC[y(0^+) - y(0^-)] = 1 \]

\[ RC[y(0^+) - 0] = 1 \]

\[ RCy(0^+) = 1 \]

\[ RCKe^{0^+} = 1 \]

\[ \Rightarrow K = \frac{1}{RC} \]