From Fourier Series to Fourier Transform

$$x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{jk\omega_F t}$$

$$X[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) e^{-jk\omega_F t} dt$$

OR

$$X[k] = \frac{1}{T_F} \int_{-\frac{T_F}{2}}^{\frac{T_F}{2}} x(t) e^{-jk\omega_F t} dt$$

$$\Rightarrow X[k] = \frac{1}{T_F} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_F t} dt, \quad when \quad T_F \to \infty$$

$$\Rightarrow X[k] = \frac{1}{T_F} \boxed{\int_{-\infty}^{\infty} x(t) e^{-jk\omega_F t} dt}, \quad when \quad T_F \to \infty$$

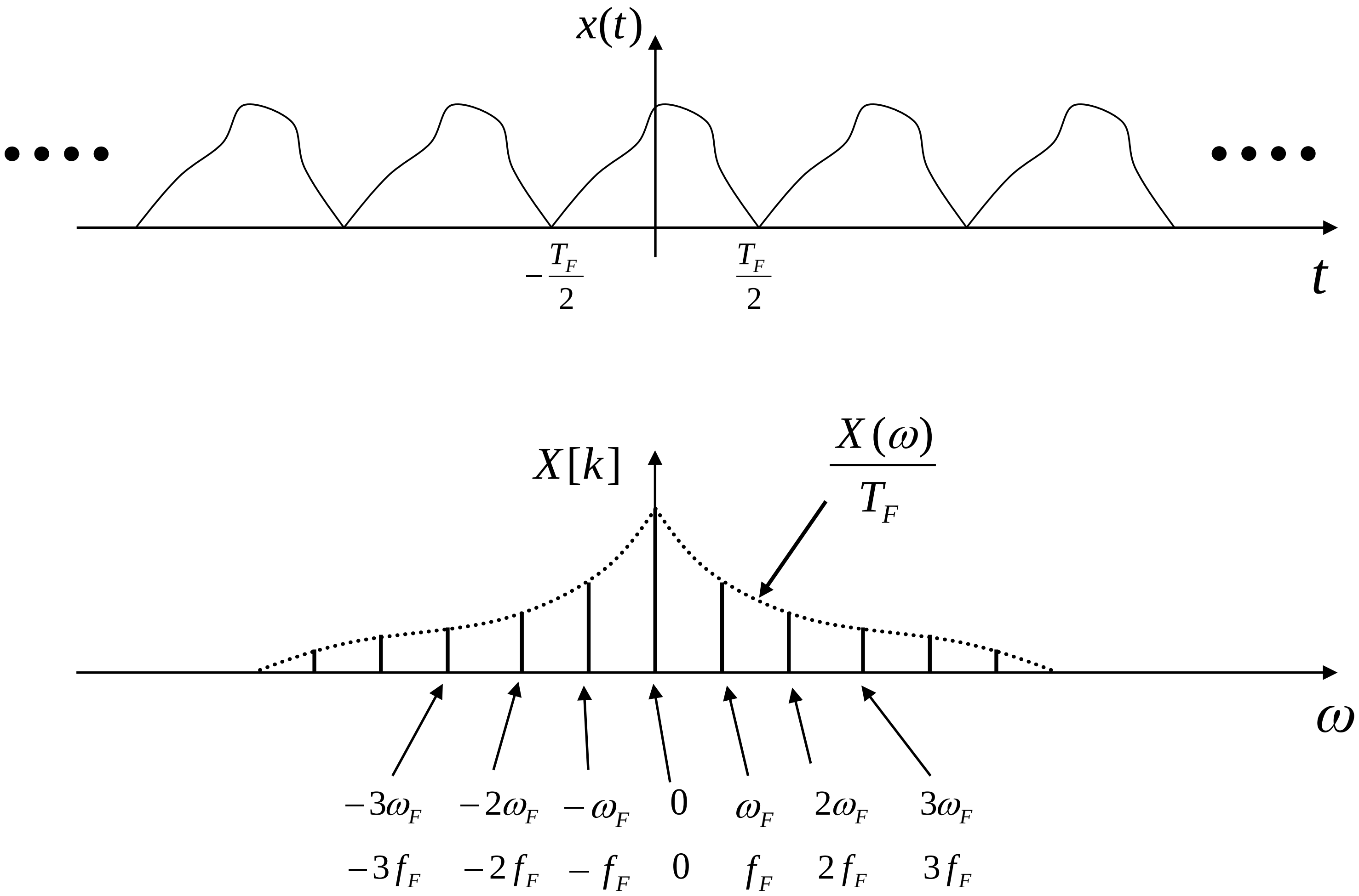
Let's Consider a function

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

We can express X[k] in terms of X(ω)

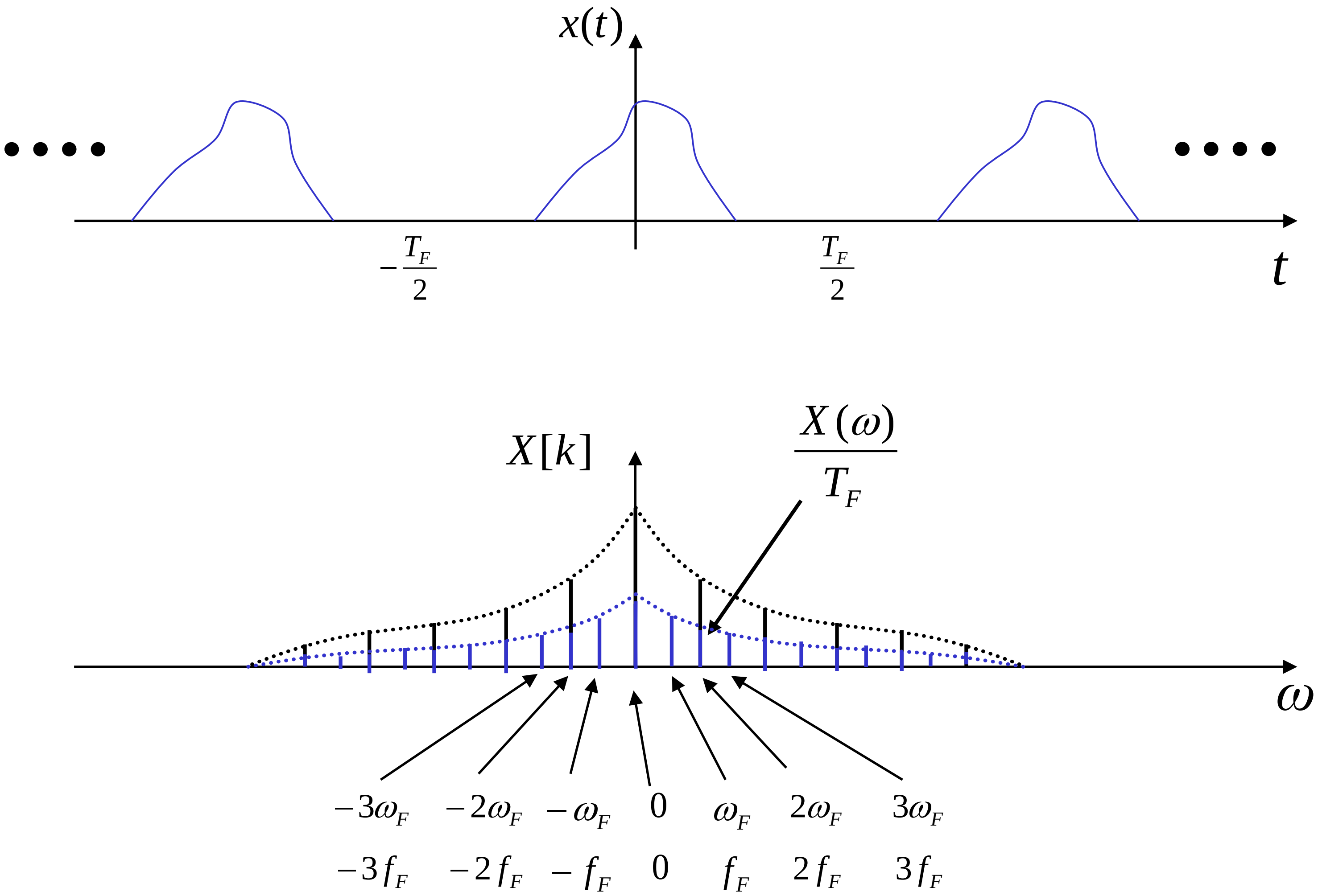
$$X[k] = \frac{1}{T_F} X(k\omega_F)$$

$1xT_F$



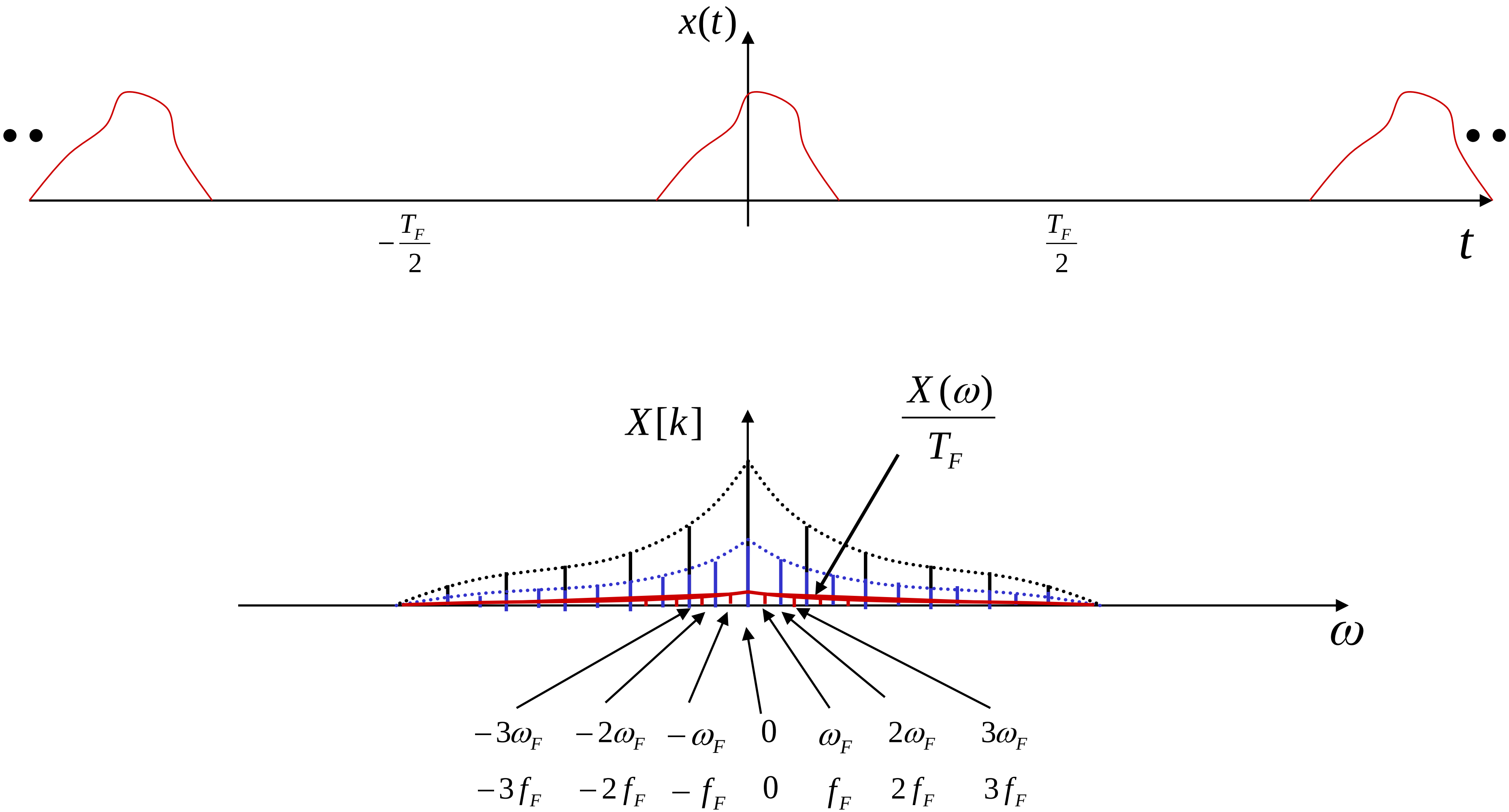
$x(t)$
$-\frac{T_F}{2}$
$\frac{T_F}{2}$
t
$X[k]$
$\frac{X(\omega)}{T_F}$
ω
$-3\omega_F$ $-2\omega_F$ $-\omega_F$ 0 ω_F $2\omega_F$ $3\omega_F$
$-3f_F$ $-2f_F$ $-f_F$ 0 f_F $2f_F$ $3f_F$

$2xT_F$



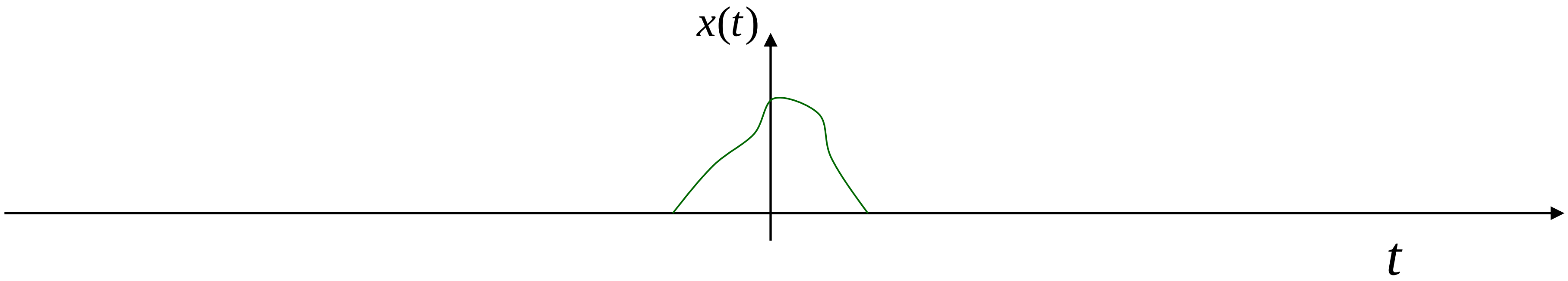
$x(t)$
t
$-\frac{T_F}{2}$
$\frac{T_F}{2}$
$X[k]$
$\frac{X(\omega)}{T_F}$
ω
$-3\omega_F$ $-2\omega_F$ $-\omega_F$ 0 ω_F $2\omega_F$ $3\omega_F$
$-3f_F$ $-2f_F$ $-f_F$ 0 f_F $2f_F$ $3f_F$

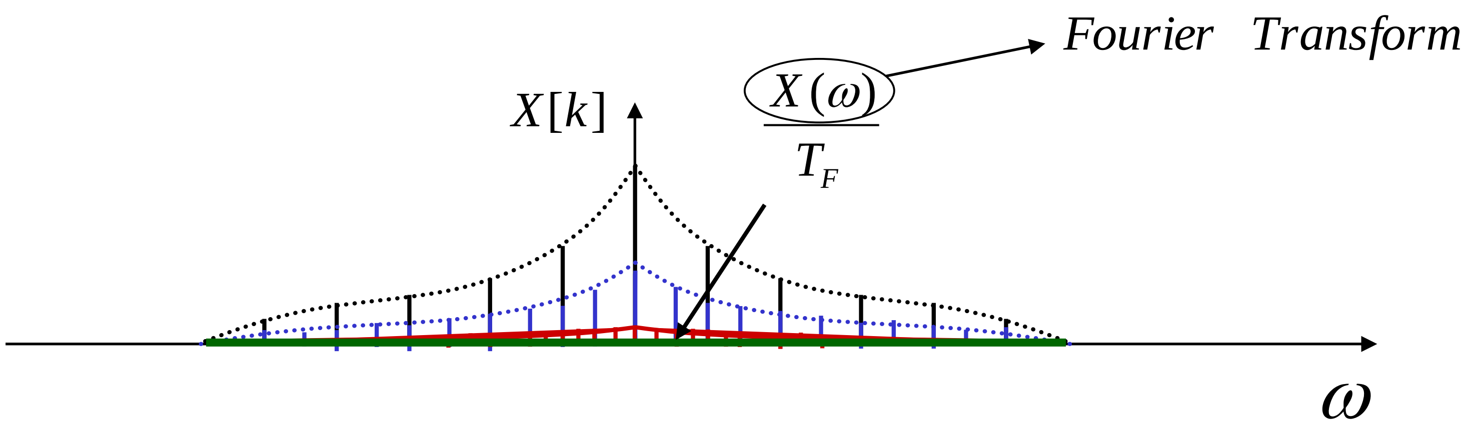
$2x2xT_F$



$x(t)$
$-\frac{T_F}{2}$
$\frac{T_F}{2}$
t
$X[k]$
$\frac{X(\omega)}{T_F}$
ω
$-3\omega_F$
$-2\omega_F$
$-\omega_F$
0
ω_F
$2\omega_F$
$3\omega_F$
$-3f_F$
$-2f_F$
$-f_F$
0
f_F
$2f_F$
$3f_F$

$$\infty x T_F$$





Fourier Transform of a Singal x(t)

$$X(\omega) = F[x(t)]$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Now Let's talk about the Inverse Fourier Transform

$$x(t) = F^{-1}[X(\omega)]$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_F t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{X(k\omega_F)}{T_F} e^{jk\omega_F t}$$

$$\lim_{T_F \to \infty} \quad \omega_F \to \Delta\omega$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{X(k\Delta\omega)}{T_F} e^{jk\Delta\omega t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{X(k\Delta\omega)\Delta\omega}{2\pi} e^{jk\Delta\omega t}$$

$$T_F = \frac{2\pi}{\omega_F} = \frac{2\pi}{\Delta\omega}$$

$$x(t)=\sum_{k=-\infty}^{\infty}\frac{X(k\Delta\omega)\Delta\omega}{2\pi}e^{jk\Delta\omega t}\qquad T_F=\frac{2\pi}{\omega_F}=\frac{2\pi}{\Delta\omega}$$

$$x(t)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}X(k\Delta\omega)\Delta\omega e^{jk\Delta\omega t}$$

$$x(t)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}X(k\Delta\omega)e^{jk\Delta\omega t}\Delta\omega$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$$

$$x(t)=F^{-1}[X(\omega)]$$ ***Inverse Fourier Transform***

Fourier Transform of a Signal x(t) $$X(\omega) = F[x(t)]$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Inverse Fourier Transform $$x(t) = F^{-1}[X(\omega)]$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\boxed{x(t) \Leftrightarrow X(\omega)}$$

Fourier Transform of a Signal x(t)

$$X(\omega) = F[x(t)]$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

OR

$$X(f) = F[x(t)]$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Inverse Fourier Transform

$$x(t) = F^{-1}[X(\omega)]$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

OR

$$x(t) = F^{-1}[X(f)]$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Inverse Fourier Transform

$$x(t) = F^{-1}[X(\omega)]$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(\omega)e^{j\omega t}d\omega$$

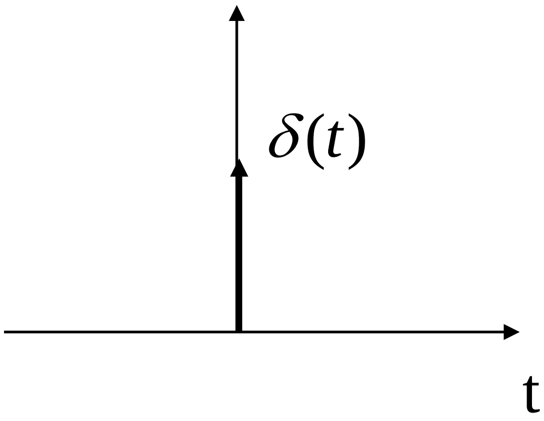
$$\frac{dx(t)}{dt} = F^{-1}[j\omega X(\omega)]$$

$$\frac{dx(t)}{dt} \Leftrightarrow j\omega X(\omega)$$

Fourier Transform of Impulse Function

Definition:

$$\begin{cases} \delta(t) = 0 \qquad t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t) = 1 \end{cases}$$

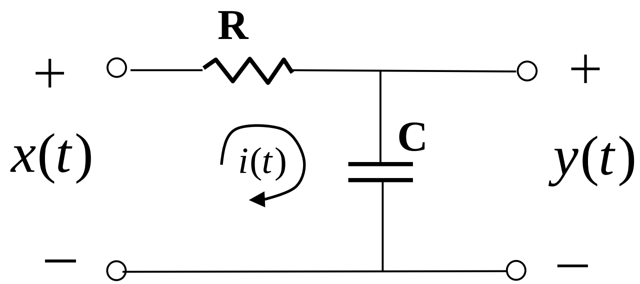


Fourier Transform of a Singal x(t)

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{0} = 1$$

Let's revisit LTI system

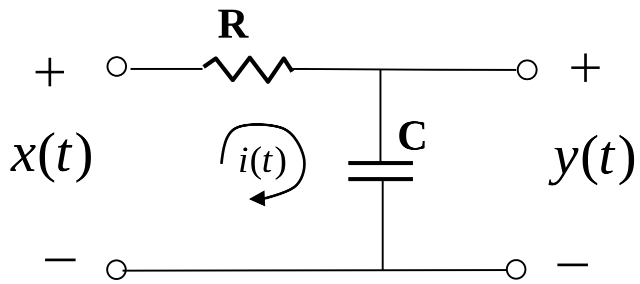


$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

$$RCj\omega Y(\omega) + Y(\omega) = X(\omega)$$

$$Y(\omega)[RCj\omega + 1] = X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{RCj\omega + 1}$$



$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{RCj\omega + 1}$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega RC}$$

When the input, x(t) is unit impulse function, the output is h(t)

$$\frac{H(\omega)}{1} = \frac{1}{1 + j\omega RC}$$

$$\Rightarrow H(\omega) = \frac{1}{1 + j\omega RC}$$

Some Examples of Fourier Transform

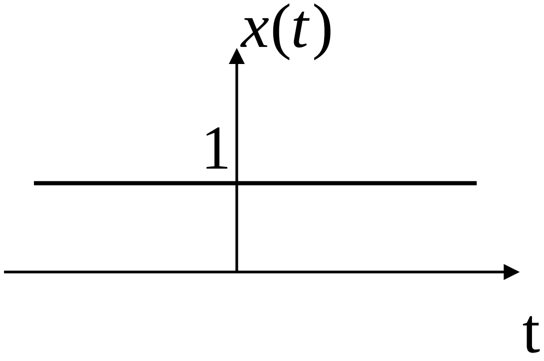
1. *An impulse function*
2. *A constant function (via inverse transform)*
3. *Complex exponential function (via inverse transform)*
4. *Sinosoidal Function*
5. *Gate Function*

Fourier Transform of a Constant Function

Fourier Transform of a Singal x(t)

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dt \quad = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-\infty}^{\infty} = 0 + \infty = \infty$$

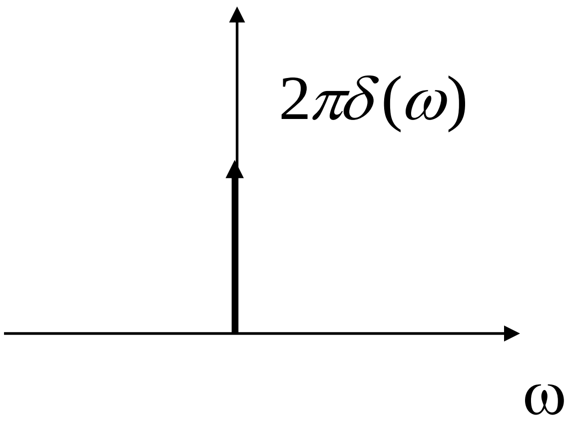


Let's try indirectly – let's find the Inverse Fourier Transform of δ(ω)

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\Rightarrow F^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{0} = \frac{1}{2\pi}$$

$$\Rightarrow F[\frac{1}{2\pi}] = \delta(\omega) \qquad \Rightarrow F[1] = 2\pi\delta(\omega)$$



Fourier Transform of a Complex Exponential

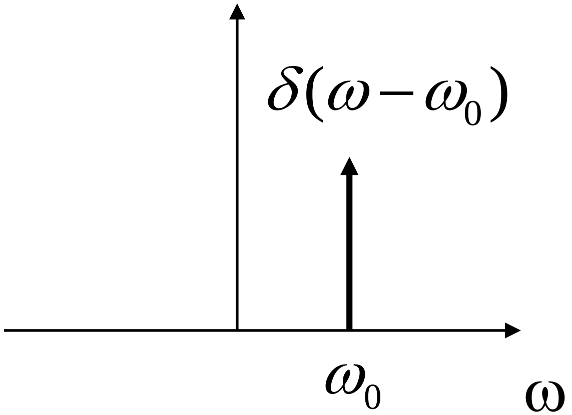
Let's find the Inverse Fourier Transform of $\delta(\omega-\omega_0)$

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$\Rightarrow F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

$$\Rightarrow F[\frac{e^{j\omega_0 t}}{2\pi}] = \delta(\omega-\omega_0)$$

$$\Rightarrow F[e^{j\omega_0 t}] = 2\pi\delta(\omega-\omega_0)$$

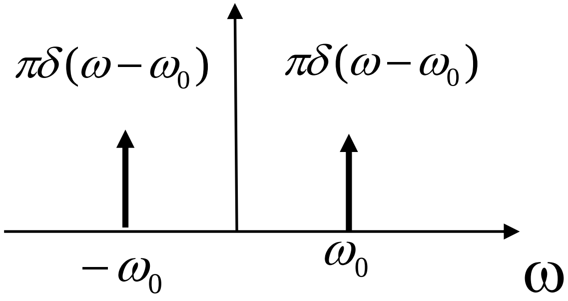


Fourier Transform of a Sinusoidal Signal

$$x(t) = \cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\Rightarrow F[\cos(\omega_0 t)] = F[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}] = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$= \pi[\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)]$$



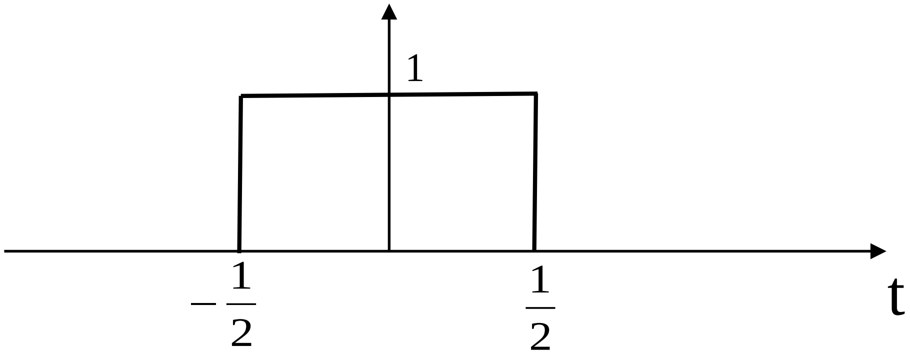
$$x(t) = \sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\Rightarrow F[\sin(\omega_0 t)] = F[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}] = -j\pi\delta(\omega - \omega_0) + \pi j\delta(\omega + \omega_0)$$

$$= j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

Fourier Transform of a Rectangular Function

$$rect(t) = \begin{cases} 1 & |t| < 1/2 \\ 1/2 & |t| = 1/2 \\ 0 & |t| > 1/2 \end{cases}$$



$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\Rightarrow F[rect(t)] = \int_{-1/2}^{1/2} e^{-j\omega t} dt \quad = \left| \frac{e^{-j\omega t}}{-j\omega} \right|_{-1/2}^{1/2} = \frac{e^{-j\omega/2}}{-j\omega} - \frac{e^{j\omega/2}}{-j\omega}$$

$$= \frac{e^{j\omega/2}}{j\omega} - \frac{e^{-j\omega/2}}{j\omega} = \frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega}$$

$$F[rect(t)] = \frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega}$$

$$= \frac{2}{\omega} \frac{[e^{j\omega/2} - e^{-j\omega/2}]}{2j}$$

$$= \frac{2}{\omega} \sin(\omega/2) = \frac{\sin(\omega/2)}{(\omega/2)}$$

Remember?

$$\sin c(t) = \frac{\sin(\pi t)}{\pi t}$$

or

$$\sin c(\omega) = \frac{\sin(\pi\omega)}{\pi\omega}$$

$$= \sin c(\omega/2\pi)$$

Some Properties of Fourier Transform

1. ***Linearity***
2. ***Symmetry***
3. ***Scaling***
4. ***Time Shifting***
5. ***Frequency Shifting***
6. ***Time Differentiation***
7. ***Convolution***

Linearity

$$k_1x_1(t)+k_2x_2(t) \Leftrightarrow k_1X_1(\omega)+k_2X_2(\omega)$$

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\Rightarrow F[kx(t)] = \int_{-\infty}^{\infty} kx(t)e^{-j\omega t}dt$$

$$= k\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= kX(\omega)$$

Symmetry

$$x(t) \Leftrightarrow X(\omega)$$
$$\Downarrow$$
$$X(t) \Leftrightarrow 2\pi x(-\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\alpha) e^{j\alpha t} d\alpha$$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(\alpha) e^{j\alpha t} d\alpha$$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\alpha) e^{-j\alpha t} d\alpha$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(\alpha) e^{-j\alpha\omega} d\alpha$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-jt\omega} dt$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

$$\Rightarrow X(t) \Leftrightarrow 2\pi x(-\omega)$$

Scaling

$$x(t) \Leftrightarrow X(\omega)$$

$$\Downarrow$$

$$x(at) \Leftrightarrow \frac{1}{|a|} X(\frac{\omega}{a})$$

Let's assume 'a' is positive

$$F[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

Let's make a variable change $at = \alpha \Rightarrow adt = d\alpha$

$$\Rightarrow F[x(at)] = \frac{1}{a} \int_{-\infty}^{\infty} x(\alpha) e^{-j\omega\alpha/a} d\alpha$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\alpha) e^{-j(\omega/a)\alpha} d\alpha = \frac{1}{a} X(\frac{\omega}{a})$$

Time Shifting – linear phase shift

$$x(t-t_0) \Leftrightarrow X(\omega)e^{-j\omega t_0}$$

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt$$

Let's make a variable change $t - t_0 = \alpha \Rightarrow dt = d\alpha$

$$\Rightarrow F[x(t-t_0)] = \int_{-\infty}^{\infty} x(\alpha)e^{-j\omega(\alpha+t_0)}d\alpha$$

$$= \int_{-\infty}^{\infty} x(\alpha)e^{-j\omega\alpha}e^{-j\omega t_0}d\alpha$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\alpha)e^{-j\omega\alpha}d\alpha = e^{-j\omega t_0}X(\omega)$$

Frequency Shifting - modulation

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(\omega-\omega_0)$$

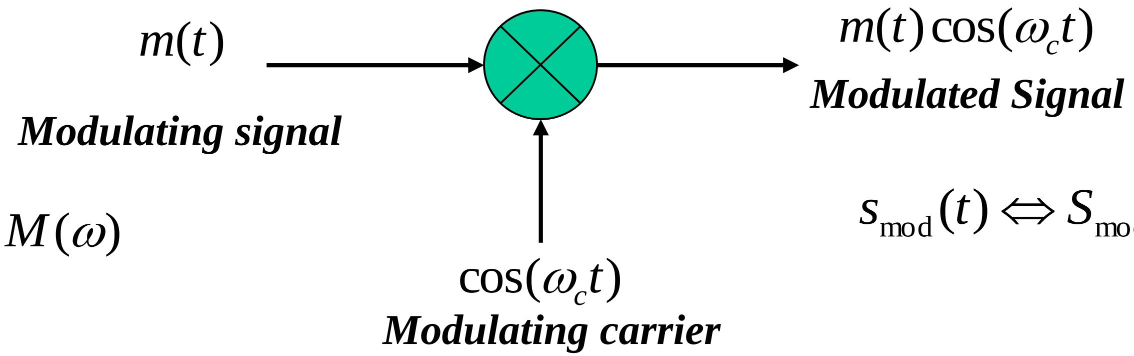
$$F[x(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j\omega t + j\omega_0 t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt$$

$$= X(\omega-\omega_0)$$

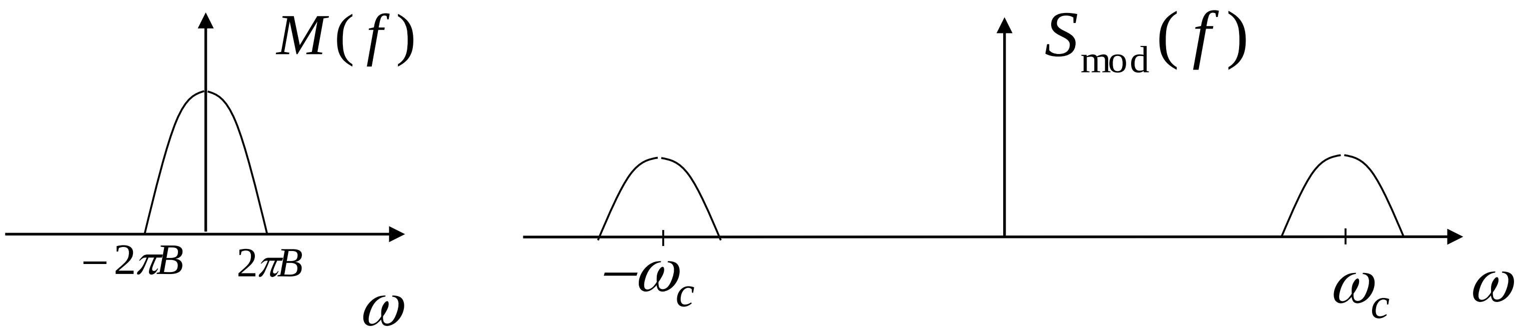
Amplitude Modulation



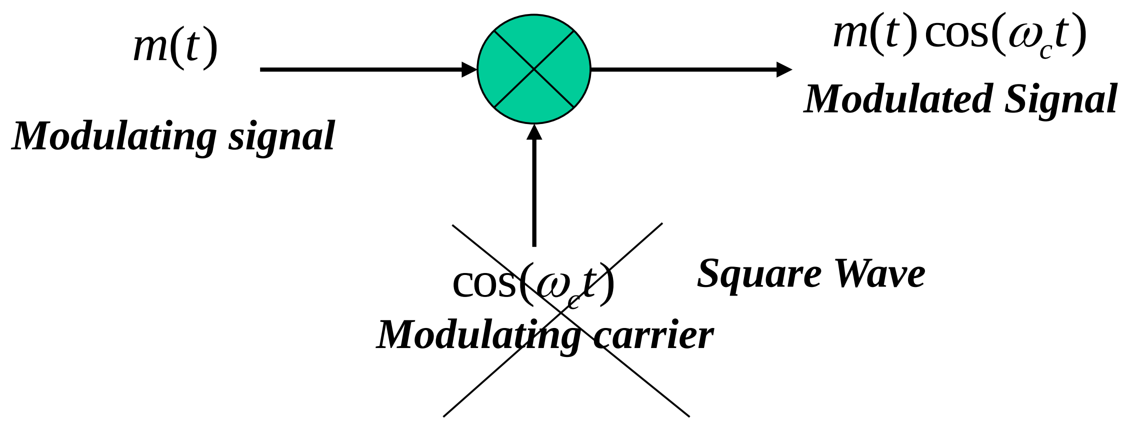
$m(t) \Leftrightarrow M(\omega)$

$s_{\text{mod}}(t) \Leftrightarrow S_{\text{mod}}(\omega)$

$$s_{\text{mod}}(t) = m(t)\cos(\omega_c t) \Leftrightarrow \frac{1}{2}[M(\omega+\omega_c) + M(\omega-\omega_c)]$$

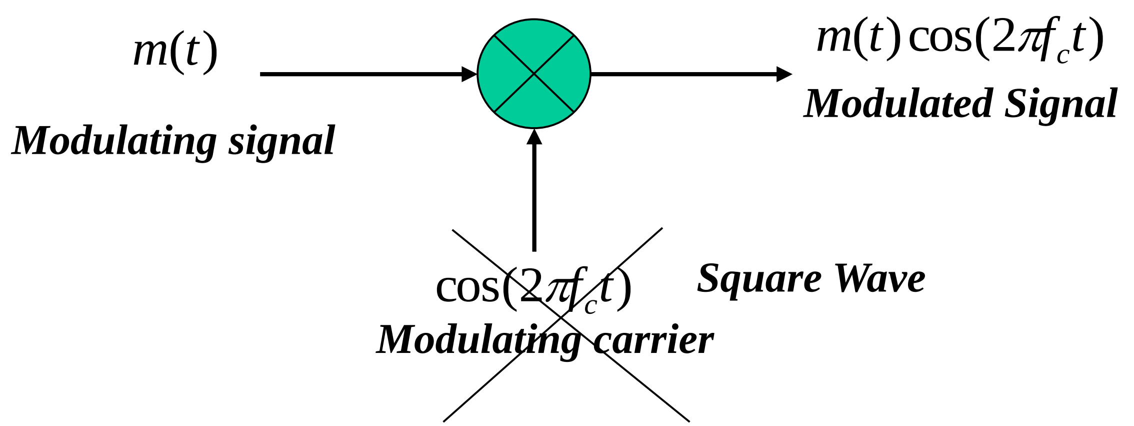


Amplitude Modulation – Another Method

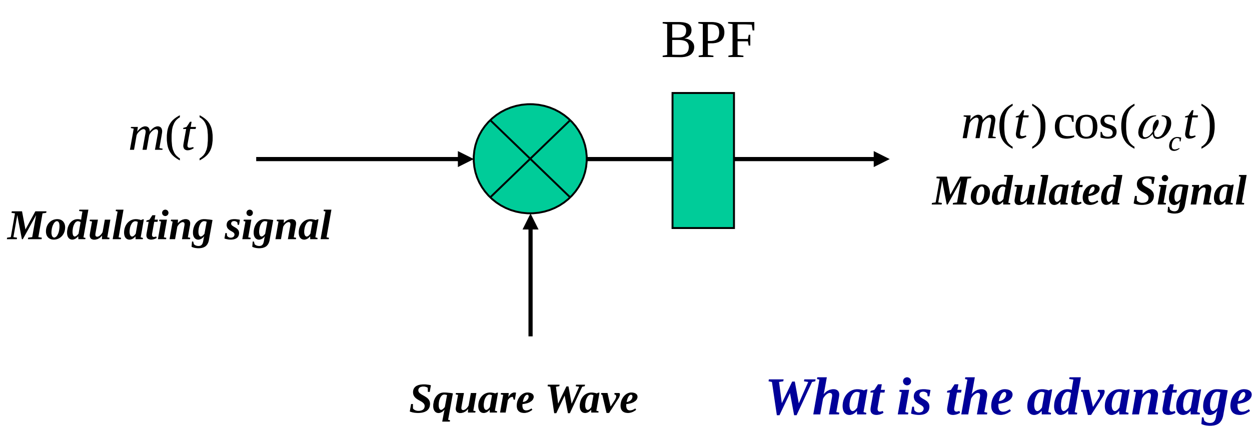


Can we do it with square wave? If so, what else do we need?

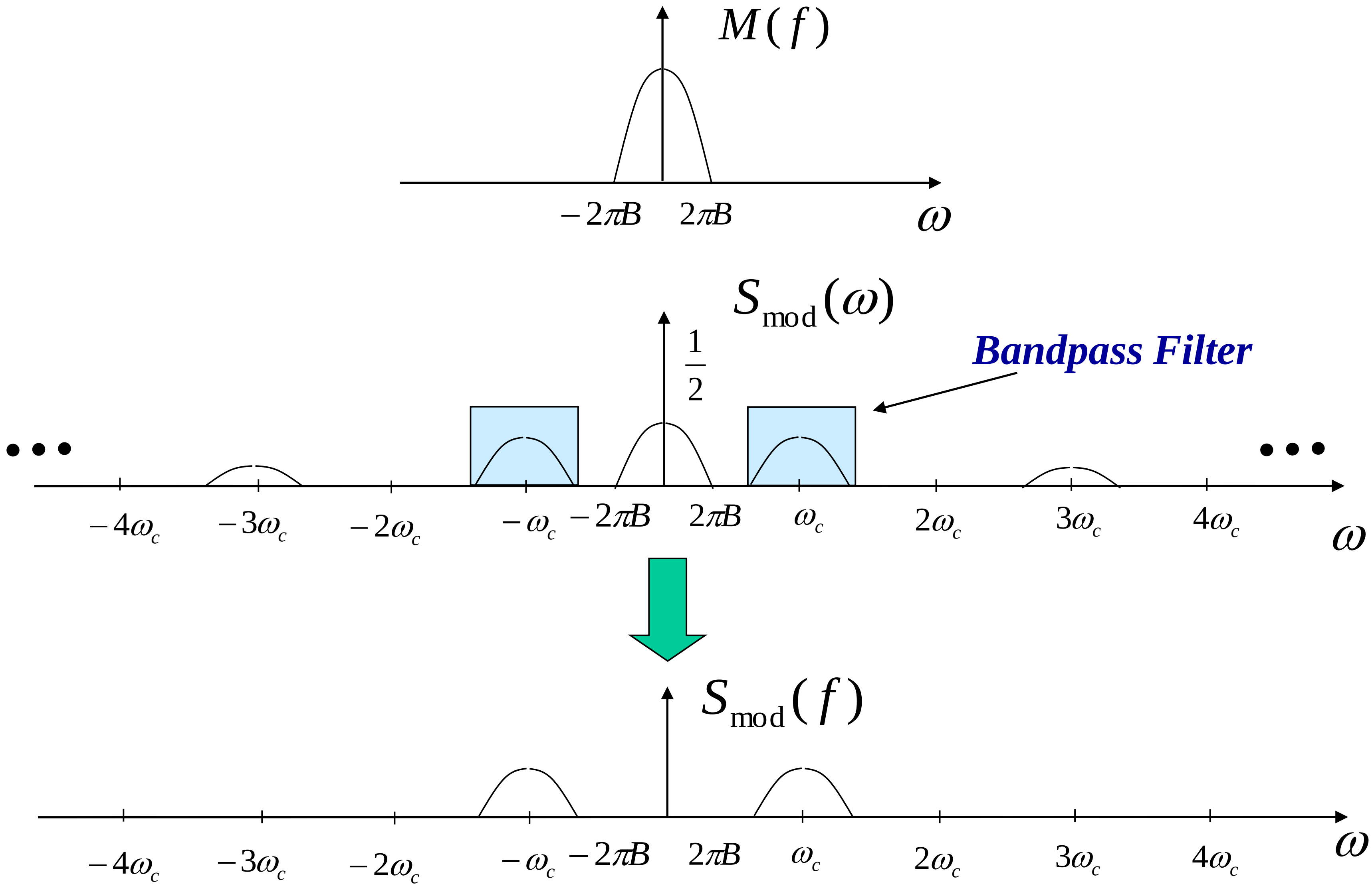
Amplitude Modulation – Another Method



Can we do it with square wave? If so, what else do we need?

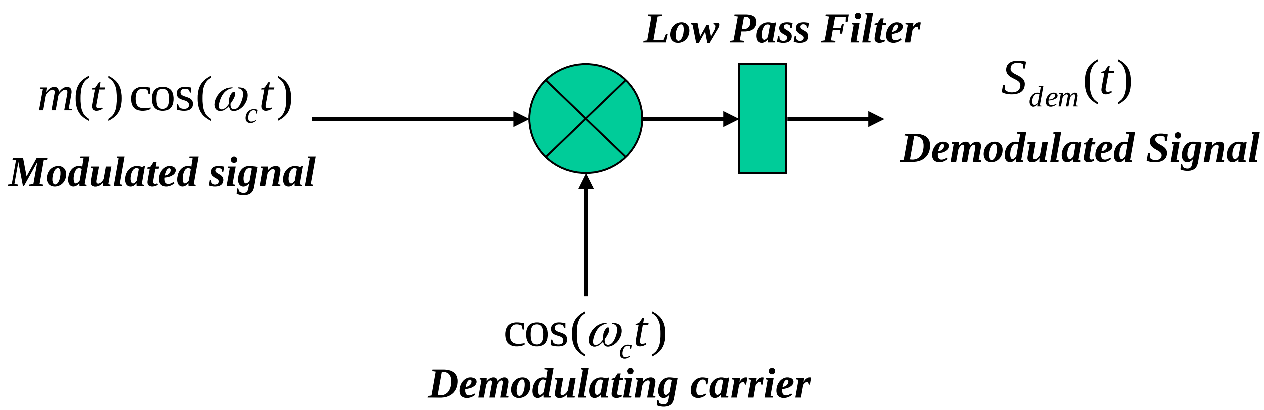


What is the advantage of doing so?



Advantage: A higher multiple of carrier frequency can be chosen

Amplitude Demodulation – Synchronous



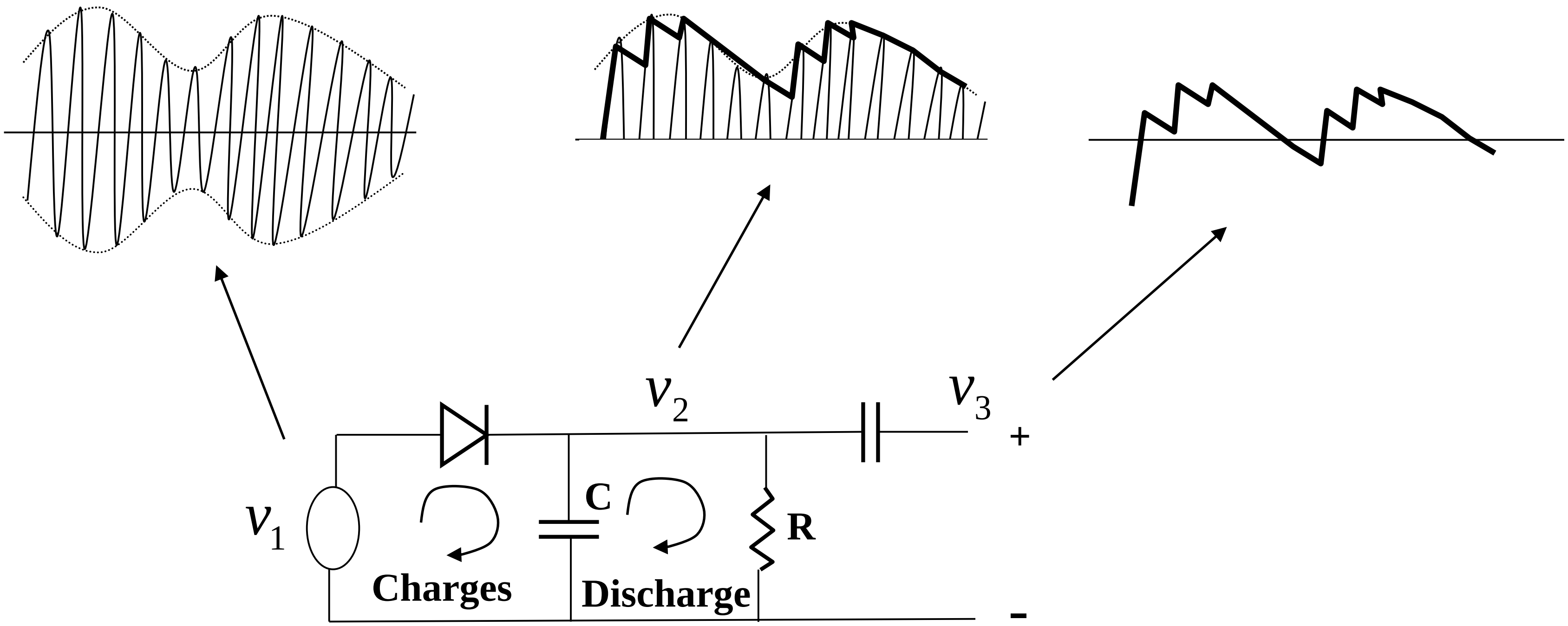
Demodulation:

$$S_{dem}(t) = m(t)\cos^2(\omega_c t)$$

$$= \frac{1}{2}[m(t) + m(t)\cos(2\omega_c t)]$$

$$S_{dem}(\omega) \Leftrightarrow \frac{1}{2}M(\omega) + \frac{1}{4}[M(\omega + 2\omega_c) + M(\omega - 2\omega_c)]$$

Amplitude Demodulation – Envelop Detection



Time Constant = RC

C should charge quick enough

$$RC > \frac{1}{\omega_c}$$

C should decay slow enough

$$RC < \frac{1}{2\pi B}$$

Time Differentiation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(\omega)e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = F^{-1}[j\omega X(\omega)]$$

$$\frac{dx(t)}{dt} \Leftrightarrow j\omega X(\omega)$$

Convolution $$x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau$$

If $x(t)\Leftrightarrow X(\omega)$ ***and*** $y(t)\Leftrightarrow Y(\omega)$

$$\Downarrow$$

Then $$x(t)*y(t)\Leftrightarrow X(\omega)Y(\omega)$$

$$F[x(t)*y(t)]=\int_{-\infty}^{\infty}e^{-j\omega t}[\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau]dt$$

$$=\int_{-\infty}^{\infty}x(\tau)[\int_{-\infty}^{\infty}e^{-j\omega t}y(t-\tau)dt]d\tau$$

$$=\int_{-\infty}^{\infty}x(\tau)[Y(\omega)e^{-j\omega\tau}]d\tau$$

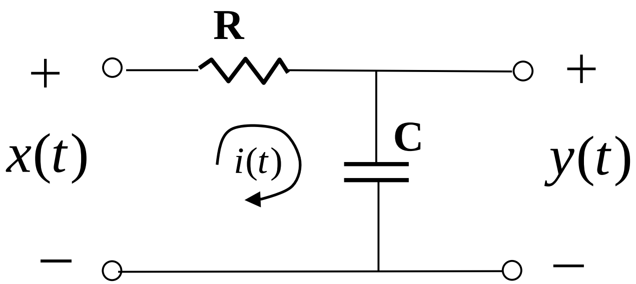
$$F[x(t)*y(t)] = \int_{-\infty}^{\infty} x(\tau)[Y(\omega)e^{-j\omega\tau}]d\tau$$

$$= Y(\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$$

$$= Y(\omega)X(\omega)$$

$$\Rightarrow x(t)*y(t) \Leftrightarrow X(\omega)Y(\omega)$$

Let's revisit LTI system - again



$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

$$RCj\omega Y(\omega) + Y(\omega) = X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega RC} = H(\omega)$$

$$\Rightarrow Y(\omega) = X(\omega)H(\omega)$$

$$but \quad we \quad know \quad y(t) = x(t) * h(t)$$

Some Properties of Fourier Transform

$$x(t) = rect(t)$$

$$X(\omega) = \sin c(\omega / 2\pi) \qquad X(f) = \sin c(f)$$

Find the Fourier Transform of the following functions

$$x(t-2), \quad 2x(t), \quad x(2t)$$

Some Properties of Fourier Transform

$$x(t) = rect(t)$$

$$X(\omega) = \sin c(\omega / 2\pi) \qquad\qquad X(f) = \sin c(f)$$

Find the Fourier Transform of the following functions

$$x(t-2), \quad 2x(t), \quad x(2t)$$

$$x(t-2) \leftrightarrow X(\omega)e^{-j2\omega} = \sin c(\frac{\omega}{2\pi})e^{-j2\omega}$$

Some Properties of Fourier Transform

$$x(t) = rect(t)$$

$$X(\omega) = \sin c(\omega / 2\pi) \qquad X(f) = \sin c(f)$$

Find the Fourier Transform of the following functions

$$x(t-2), \quad 2x(t), \quad x(2t)$$

$$x(t-2) \leftrightarrow X(\omega)e^{-j2\omega} = \sin c(\frac{\omega}{2\pi})e^{-j2\omega}$$

$$2x(t) \leftrightarrow 2X(\omega) = 2\sin c(\frac{\omega}{2\pi})$$

Some Properties of Fourier Transform

$$x(t) = rect(t)$$

$$X(\omega) = \sin c(\omega / 2\pi) \qquad X(f) = \sin c(f)$$

Find the Fourier Transform of the following functions

$$x(t-2), \quad 2x(t), \quad x(2t)$$

$$x(t-2) \leftrightarrow X(\omega)e^{-j2\omega} = \sin c(\frac{\omega}{2\pi})e^{-j2\omega}$$

$$2x(t) \leftrightarrow 2X(\omega) = 2\sin c(\frac{\omega}{2\pi})$$

$$x(2t) \leftrightarrow \frac{1}{2}X(\frac{\omega}{2}) = \frac{1}{2}\sin c(\frac{\omega}{4\pi})$$

Some Properties of Fourier Transform

$$x(t) = rect(t)$$

$$X(\omega) = \sin c(\omega / 2\pi) \qquad X(f) = \sin c(f)$$

Find the Fourier Transform of the following functions

$$x(t-2), \quad 2x(t), \quad x(2t)$$

$$x(t-2) \leftrightarrow X(\omega)e^{-j2\omega} = \sin c(\frac{\omega}{2\pi})e^{-j2\omega}$$

$$2x(t) \leftrightarrow 2X(\omega) = 2\sin c(\frac{\omega}{2\pi})$$

$$x(2t) \leftrightarrow \frac{1}{2}X(\frac{\omega}{2}) = \frac{1}{2}\sin c(\frac{\omega}{4\pi})$$

How about FT of

$$2x(2t-2)$$

$$\updownarrow$$

$$\sin c(\frac{\omega}{4\pi})e^{-j\omega}$$

Time For Discrete Time Fourier Series

CT Exponential Fourier Series

$$x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t}$$

where $$f_F = \frac{1}{T_F}$$

$$X[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) e^{-j2\pi(kf_F)t} dt$$

DT Exponential Fourier Series

$$x[n] = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kF_F)n}$$

where $$F_F = \frac{1}{N_F}$$

$$X[k] = \frac{1}{N_F} \sum_{n=n_0}^{n=n_0+N_F-1} x[n] e^{-j2\pi(kF_F)n}$$

One Important Difference

CT Exponential Fourier Series

$$x(t)=\sum_{k=-\infty}^{k=\infty} X[k]e^{j2\pi(kf_F)t}$$

where $f_F=\dfrac{1}{T_F}$

Basis function $e^{j2\pi(kf_F)t}$

$$e^{j2\pi(k+T_F)f_F t}=e^{j2\pi(kf_F)t}e^{j2\pi(T_F f_F)t}$$

$$=e^{j2\pi(kf_F)t}e^{j2\pi t}$$

$$=e^{j2\pi(kf_F)t}[\cos(2\pi t)+j\sin(2\pi t)]$$

$$\neq e^{j2\pi(kf_F)t}$$ **for all t**

DT Exponential Fourier Series

$$x[n]=\sum_{k=-\infty}^{k=\infty} X[k]e^{j2\pi(kF_F)n}$$

where $F_F=\dfrac{1}{N_F}$

Basis function $e^{j2\pi(kF_F)n}$

$$e^{j2\pi(k+N_F)F_F n}=e^{j2\pi(kF_F)n}e^{j2\pi(N_F F_F)n}$$

$$=e^{j2\pi(kF_F)n}e^{j2\pi n}$$

$$=e^{j2\pi(kF_F)n}[\cos(2\pi n)+j\sin(2\pi n)]$$

$$=e^{j2\pi(kF_F)n}$$ **for all n**

One Important Difference … cont.

CT Exponential Fourier Series

$$x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t}$$

where $f_F = \frac{1}{T_F}$

Basis function $e^{j2\pi(kf_F)t}$

Still need infinite number of exponential functions

$$\Rightarrow x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t}$$

DT Exponential Fourier Series

$$x[n] = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kF_F)n}$$

where $F_F = \frac{1}{N_F}$

Basis function $e^{j2\pi(kF_F)n}$

Need only N_F exponential Functions

$$\Rightarrow x[n] = \sum_{k=k_0}^{k=k_0+N_F-1} X[k] e^{j2\pi(kF_F)n}$$

DTFS vs. CTFS

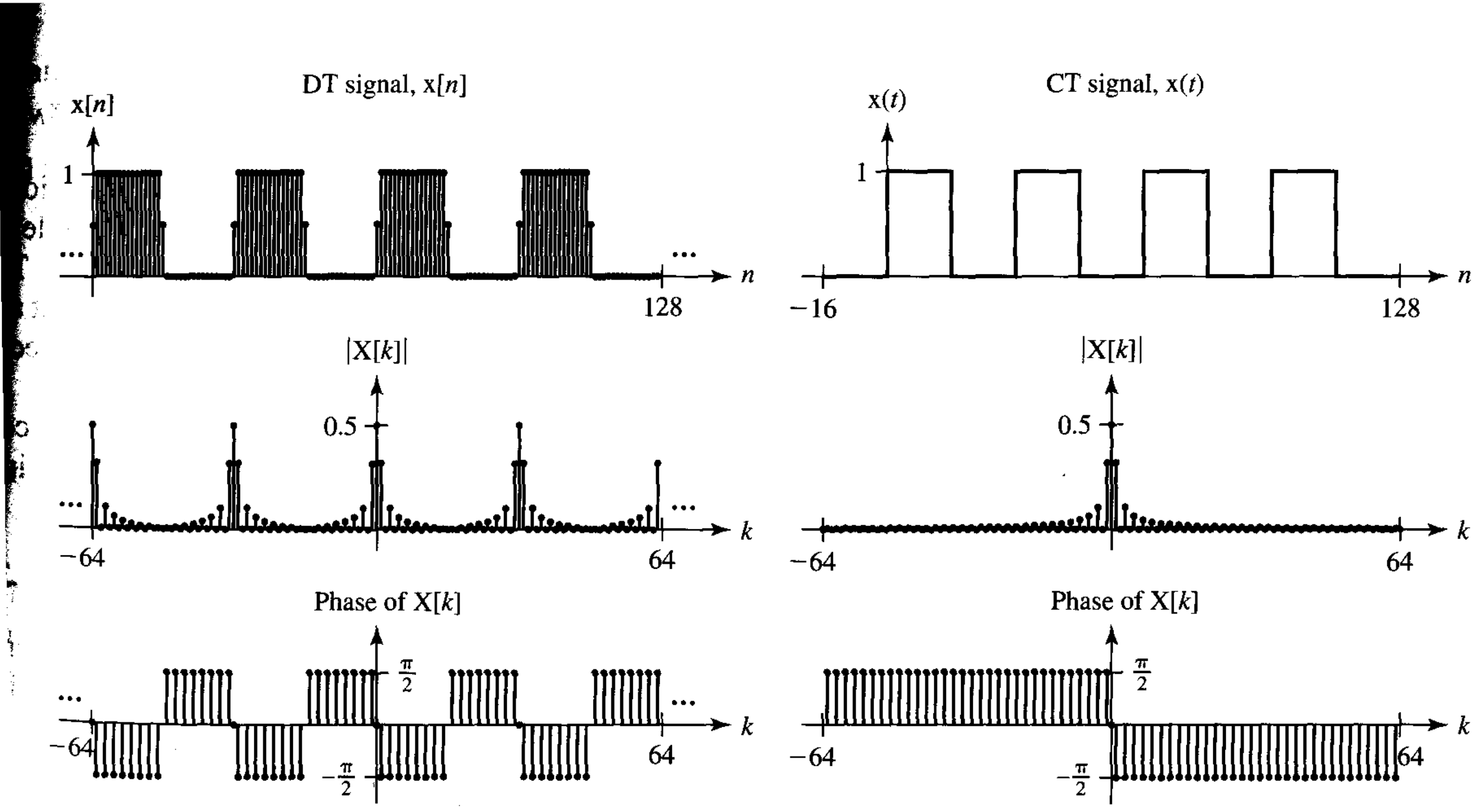


Figure 4.50
A DT signal with its DTFS harmonic function and the corresponding CT signal with its CTFS harmonic function.

Transition from DT Fourier Series to DT Fourier Transform

$$X[k] = \frac{1}{N_F} \sum_{n=n_0}^{n=n_0+N_F-1} x[n] e^{-j2\pi(kF_F)n}$$

$$X[k] = \frac{1}{N_F} \sum_{n=-N_F/2}^{n=(N_F-1)/2} x[n] e^{-j2\pi(kF_F)n}$$

$$X[k] = \frac{1}{N_F} \sum_{n=-\infty}^{n=\infty} x[n] e^{-j2\pi(kF_F)n}$$ ***When*** $N_F \to \infty$

Let's define a function

$$X(F) = \sum_{n=-\infty}^{n=\infty} x[n] e^{-j2\pi Fn} \qquad \Rightarrow X[k] = \frac{1}{N_F} X(kF_F)$$

Transition from DT Fourier Series to DT Fourier Transform

$$X[k]=\frac{1}{N_F}\sum_{n=n_0}^{n=n_0+N_F-1}x[n]e^{-j2\pi(kF_F)n}$$

$$X[k]=\frac{1}{N_F}\sum_{n=-(N_F-1)/2}^{n=(N_F-1)/2}x[n]e^{-j2\pi(kF_F)n}$$

$$X[k]=\frac{1}{N_F}\sum_{n=-\infty}^{n=\infty}x[n]e^{-j2\pi(kF_F)n}$$ ***When*** $N_F\rightarrow\infty$

DTFT

Let's define a function

$$\boxed{X(F)=\sum_{n=-\infty}^{n=\infty}x[n]e^{-j2\pi Fn}}$$

$$\Rightarrow X[k]=\frac{1}{N_F}X(kF_F)$$

Inverse Discrete Fourier Transform

$$x[n] = \sum_{k=k_0}^{k=k_0+N_F-1} X[k] e^{j2\pi(kF_F)n}$$

$$x[n] = \sum_{k=0}^{k=N_F-1} X[k] e^{j2\pi(kF_F)n}$$

$$x[n] = \sum_{k=0}^{k=N_F-1)} \frac{X(kF_F)}{N_F} e^{j2\pi(kF_F)n}$$

$$x[n] = \sum_{k=0}^{k=N_F-1} \frac{X(k\Delta F)}{1/\Delta F} e^{j2\pi(k\Delta F)n}$$

When $N_F \to \infty$

$$x[n]=\sum_{k=0}^{k=N_F-1}\frac{X(k\Delta F)}{1/\Delta F}e^{j2\pi(k\Delta F)n}$$

When $N_F \to \infty$

$$x[n]=\sum_{k=0}^{k=N_F-1}X(k\Delta F)\Delta F e^{j2\pi(k\Delta F)n}$$

$$k\Delta F \quad changes \quad from \quad 0 \to 1$$

$$\Rightarrow x[n]=\int_{1} X(F)e^{j2\pi Fn}dF$$

DT Inverse Fourier Transform

Discrete Time
Fourier Transform

$$X(F)=\sum_{n=-\infty}^{n=\infty} x[n]e^{-j2\pi Fn}$$

OR

$$X(\Omega)=\sum_{n=-\infty}^{n=\infty} x[n]e^{-j\Omega n}$$

Discrete Time
Inverse Fourier Transform

$$x[n]=\int_{1} X(F)e^{j2\pi Fn}dF$$

OR

$$x[n]=\frac{1}{2\pi}\int_{2\pi} X(F)e^{j\Omega n}d\Omega$$

Where $2\pi F=\Omega$

Some Properties of Discrete Fourier Transform

1. Linearity
2. Scaling
3. Time Shifting
4. Frequency Shifting
5. Time Differencing
6. Convolution

Relationship between Time and Frequency Domains

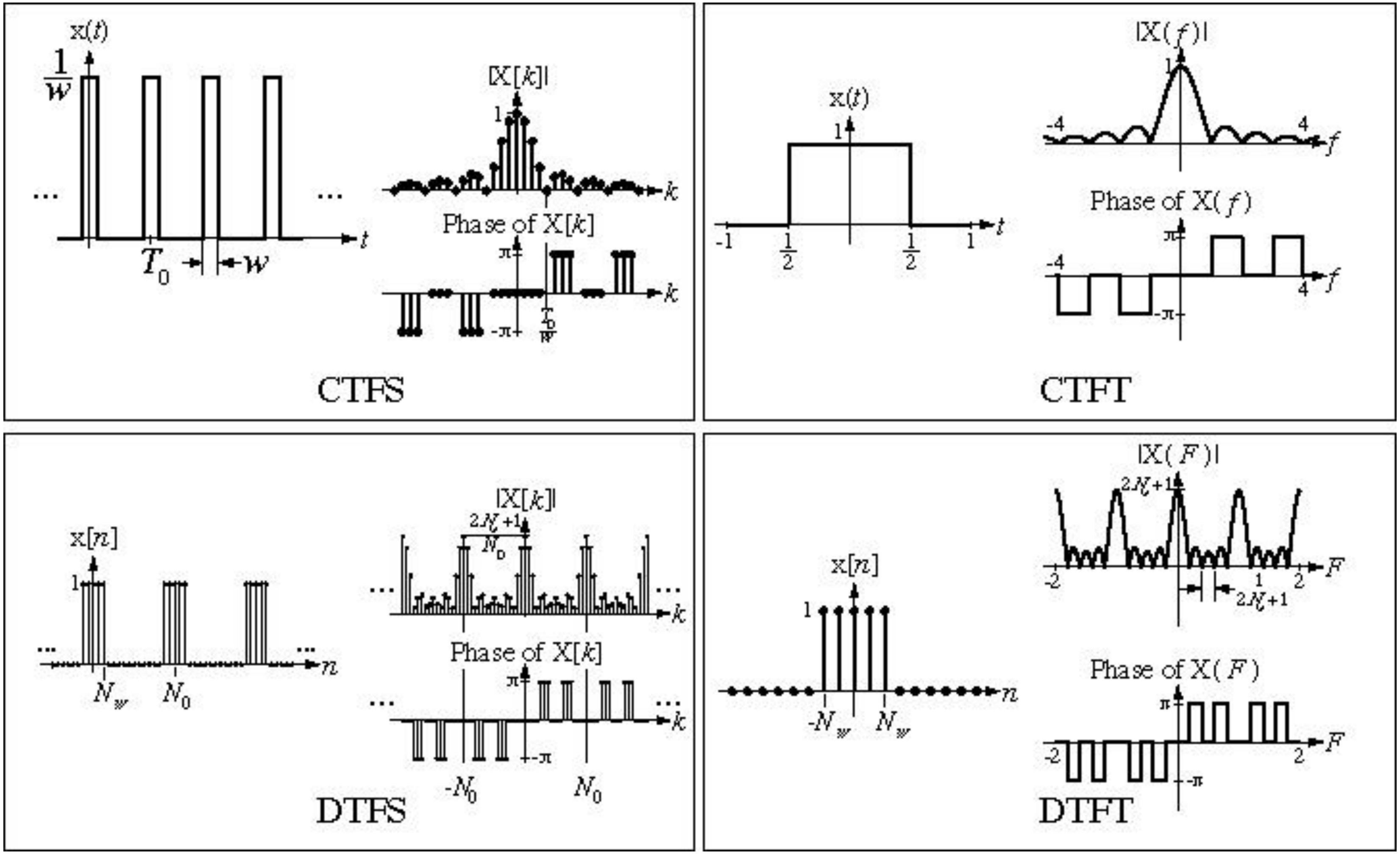
	Discrete Frequency	***Continuous Frequency***
Continuous Time	CTFS	CTFT
Discrete Time	DTFS	DTFT

With Examples

Discrete Frequency | **Continuous Frequency**

CT

DT



Parseval's Theorem in Continuous Frequency

Energy of CT Signals

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Energy of DT Signals

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \int_{1} |X(F)|^2 \, dF$$

$$= \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 \, d\Omega$$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t) x^*(t) dt$$

$$= \int_{-\infty}^{\infty} x(t) [\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) [\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) X(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Parseval's Theorem in Discrete Frequency

Power of Periodic CT Signals

$$P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt$$

$$= \sum_{k=-\infty}^{\infty} |X[k]|^2$$

Power of Periodic DT Signals

$$P_x = \frac{1}{N_0} \sum_{n=<N_0>} |x[n]|^2$$

$$= \sum_{k=<N_0>} |X[k]|^2$$

$$x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t}$$

Energy of each of the above frequency component over T_0

$$E_{x,T_0,X[k]} = \int_{T_0} \left| X[k] e^{j2\pi(kf_F)t} \right|^2 dt$$

$$= \int_{T_0} \left| X[k] \right|^2 \left| e^{j2\pi(kf_F)t} \right|^2 dt$$

$$= \int_{T_0} \left| X[k] \right|^2 dt = \left| X[k] \right|^2 T_0$$

$$x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t}$$

$$= X[0] + X[1]e^{j2\pi(f_F)t} + X[2]e^{j2\pi(2f_F)t} + \ldots$$

$$+ X[-1]e^{j2\pi(-f_F)t} + X[-2]e^{j2\pi(-2f_F)t} + \ldots$$

From conservation of energy principle

$$E_{x,T_0} = T_0[|X[0]|^2 + |X[1]|^2 + |X[2]|^2 + \ldots$$

$$+ |X[-1]|^2 + |X[-2]|^2 + \ldots]$$

$$\Rightarrow P_x = \frac{1}{T_0} E_{x,T_0} = [|X[0]|^2 + |X[1]|^2 + |X[2]|^2 + \ldots$$

$$+ |X[-1]|^2 + |X[-2]|^2 + \ldots]$$

$$\Rightarrow P_x = \frac{1}{T_0} E_{x,T_0} = [\left|X[0]\right|^2 + \left|X[1]\right|^2 + \left|X[2]\right|^2 + \ldots$$

$$+\left|X[-1]\right|^2 + \left|X[-2]\right|^2 + \ldots]$$

$$\Rightarrow P_x = \sum_{k=-\infty}^{\infty} \left|X[k]\right|^2$$

Example

Determine the power of x(t) without performing any integration

$$x(t) = A\cos(2\pi f_F t) \quad = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t}$$

$$= X[1]e^{j2\pi(f_F)t} + X[-1]e^{j2\pi(-f_F)t}$$

$$= \frac{A}{2} e^{j2\pi(f_F)t} + \frac{A}{2} e^{j2\pi(-f_F)t}$$

$$\Rightarrow P_x = \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$