Application of Combined Fourier Series Transform (Sampling Theorem)

\[ x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \leftrightarrow X[k] \]

\[ m(t) \leftrightarrow M(f) \]

Sampling Frequency

\[ f_s = \frac{1}{T_s} \]
We need Fourier Series

\[ x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \leftrightarrow X[k] \]

\[ x(t) = X[0] + \sum_{k=1}^{\infty} \left( X_c[k] \cos(2\pi kf_s t) + X_s[k] \sin(2\pi kf_s t) \right) \]

\[ X[0] = \frac{1}{T_s} \int_{t_0}^{t_0 + T_s} x(t) \, dt = \frac{1}{T_s} \int_{0}^{T_s} \delta(t) \, dt = \frac{1}{T_s} \]

\[ X_s[k] = \frac{2}{T_s} \int_{t_0}^{t_0 + T_s} x(t) \sin(2\pi kf_s t) \, dt = 0 \]
\[ X_c[k] = \frac{2}{T_s} \int_{t_0}^{t_0 + T_s} x(t) \cos(2\pi k f_s t) dt \]

\[ = \frac{2}{T_s} \int_0^{T_s} \delta(t) \cos(2\pi k f_s t) dt = \frac{2}{T_s} \cos(0) = \frac{2}{T_s} \]

Let's go back to Fourier Series representation

\[ x(t) = X[0] + \sum_{k=1}^{k=\infty} \left( X_c[k] \cos(2\pi k f_s t) + X_s[k] \sin(2\pi k f_s t) \right) \]

\[ = X[0] + \sum_{k=1}^{k=\infty} X_c[k] \cos(2\pi k f_s t) \]

\[ \Rightarrow x(t) = \frac{1}{T_s} + \frac{2}{T_s} \sum_{k=1}^{k=\infty} \cos(2\pi k f_s t) \]
What about $m(t)$? – need Fourier Transform

$B$ is the Bandwidth of $m(t)$
We now have $x(t)$ as frequency representation via Fourier Series

$$x(t) = \frac{1}{T_s} + \frac{2}{T_s} \sum_{k=1}^{\infty} \cos(2\pi kf_s t)$$

And $m(t)$ as Frequency representation via Fourier Transform

$$m(t) \leftrightarrow M(f)$$

Let's multiply $m(t)$ with $x(t)$

$$m(t)x(t) = \frac{1}{T_s} m(t) + \frac{2}{T_s} \sum_{k=1}^{\infty} m(t) \cos(2\pi kf_s t)$$

What is the frequency representation of $m(t)x(t)$?
\[ m(t)x(t) = \frac{1}{T_s} m(t) + \frac{2}{T_s} \sum_{k=1}^{\infty} m(t) \cos(2\pi kf_s t) \]

**We need to take Fourier Transform**

\[
F[m(t)x(t)] = F\left[ \frac{1}{T_s} m(t) + \frac{2}{T_s} \sum_{k=1}^{\infty} m(t) \cos(2\pi kf_s t) \right]
\]

\[
M_s(f) = \frac{1}{T_s} M(f) + \frac{2}{T_s} \sum_{k=1}^{\infty} \left[ \frac{1}{2} (M(f - kf_s) + M(f + kf_s)) \right]
\]

\[
= \frac{1}{T_s} M(f) + \frac{1}{T_s} \sum_{k=1}^{\infty} [M(f - kf_s) + M(f + kf_s)]
\]
\[
F[m(t)x(t)] = M_s(f) = \frac{1}{T_s} M(f) + \frac{1}{T_s} \sum_{k=1}^{k=\infty} [M(f - kf_s) + M(f + kf_s)]
\]
\[ M_s(f) \]

\[ f_s > 2B \]

\[ f_s < 2B \]

\[ f_s = 2B \]
\[ M_s(f) \]

\[ f_s > 2B \]

\[ f_s < 2B \]

**Sampling Theorem**

\[ f_s = 2B \]
Sampling Theorem

Sampling frequency should be at least equal to or greater than twice the bandwidth of the message signal for successful recovery of the signal from its samples.

\[ f_s \geq 2B \]

Lowpass Filter

bandwidth of lowpass filter = B Hz
A practical problem with Sampling

\[ x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \iff X[k] \]

\[ m(t) \iff M(f) \]

Not practical to generate!

Sampling Frequency

\[ f_s = \frac{1}{T_s} \]
Let’s Try Square Wave

\[ x(t) \leftrightarrow X[k] \]

Problem Resolved

\[ m_s(t) \leftrightarrow M_s(f) \]

Sampling Frequency

\[ f_s = \frac{1}{T_s} \]
We need Fourier Series

We know

\[ X[0] = \frac{1}{2} \]

\[ X_c[k] = \frac{4}{\pi k} \sin\left(\frac{\pi k}{2}\right) \]

\[ X_s[k] = 0 \]

and

\[ x(t) = X[0] + \sum_{k=1}^{\infty} \left( X_c[k] \cos(2\pi k f_s t) + X_s[k] \sin(2\pi k f_s t) \right) \]

\[ \Rightarrow x(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi k} \sin\left(\frac{\pi k}{2}\right) \cos(2\pi k f_s t) \]
We now have $x(t)$ as frequency representation via Fourier Series

$$x(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi k} \sin\left(\frac{\pi k}{2}\right) \cos(2\pi kf_s t)$$

And $m(t)$ as Frequency representation via Fourier Transform

$$m(t) \leftrightarrow M(f)$$

Let's multiply $m(t)$ with $x(t)$

$$m(t)x(t) = \frac{1}{2} m(t) + \sum_{k=1}^{\infty} \frac{4}{\pi k} \sin\left(\frac{\pi k}{2}\right) m(t) \cos(2\pi kf_s t)$$

What is the frequency representation of $m(t)x(t)$?
\[ m(t) x(t) = \frac{1}{2} m(t) + \sum_{k=1}^{k=\infty} \frac{4}{\pi k} \sin\left(\frac{\pi k}{2}\right) m(t) \cos(2\pi kf_s t) \]

We need to take Fourier Transform

\[ F[m(t) x(t)] = F\left[\frac{1}{2} m(t) + \sum_{k=1}^{k=\infty} \frac{4}{\pi k} \sin\left(\frac{\pi k}{2}\right) m(t) \cos(2\pi kf_s t)\right] \]

\[ M_s(f) = \frac{1}{2} M(f) + \sum_{k=1}^{k=\infty} \frac{4}{\pi k} \sin\left(\frac{\pi k}{2}\right) \left[ \frac{1}{2} (M(f - kf_s) + M(f + kf_s)) \right] \]

\[ = \frac{1}{2} M(f) + \sum_{k=1}^{k=\infty} \frac{2}{\pi k} \sin\left(\frac{\pi k}{2}\right) \left[ (M(f - kf_s) + M(f + kf_s)) \right] \]
\[
F[m(t)x(t)] = M_s(f) = \frac{1}{2} M(f) + \sum_{k=1}^{k=\infty} \frac{4}{\pi k} \sin([M(f - kf_s) + M(f + kf_s)]
\]

\[
M(f)
\]

\[
M_s(f)
\]

\[
-4f_s \quad -3f_s \quad -2f_s \quad -f_s \quad -B \quad B \quad f_s \quad 2f_s \quad 3f_s \quad 4f_s
\]
\[ F[m(t)x(t)] = M_s(f) = \frac{1}{2} M(f) + \sum_{k=1}^{\infty} \frac{4}{\pi k} \sin([M(f - kf_s) + M(f + kf_s)] \]

**Lowpass Filter**

**Sampling Theorem still works with practical samples**
Distortion and Linear systems

\[ x(t) \rightarrow h(t) \rightarrow y(t) \]

\( h(t) \) - Impulse Response of a Linear Time Invariant System

\[ y(t) = x(t) * h(t) \]

\[ \Rightarrow Y(\omega) = X(\omega)H(\omega) \quad h(t) \leftrightarrow H(\omega) \]
**Distortionless System**

\[ y(t) = kx(t - t_d) \]

\[ \Rightarrow Y(\omega) = kX(\omega)e^{-j\omega t_d} \]

We know \( Y(\omega) = X(\omega)H(\omega) \)

\[ \Rightarrow H(\omega) = ke^{-j\omega t_d} \]
Filters

**Lowpass Filter**

$|H(f)|$

$-f_{cut-off}$ $f_{cut-off}$

$f$

**Highpass Filter**

$|H(f)|$

$-f_{cut-off}$ $f_{cut-off}$

$f$

**Bandpass Filter**

$|H(f)|$

$-f_{center}$ $f_{center}$

$B$

$f$

**Cut-off Frequency**

Center Frequency

Bandwidth
Lowpass and Highpass combined to generate Bandpass Filter

**Lowpass Filter**

$$-f_{c,LP} \quad -f_{c,HP} \quad f_{c,HP} \quad f_{c,LP}$$

**Highpass Filter**

$$B = f_{c,LP} - f_{c,HP}$$

$$f_{\text{center}} = \frac{f_{c,LP} + f_{c,HP}}{2}$$
Lowpass and Highpass combined to generate Bandstop Filter

**Lowpass Filter**

![Diagram of Lowpass Filter]

**Highpass Filter**

![Diagram of Highpass Filter]

\[ B_{\text{stopband}} = f_{c,HP} - f_{c,LP} \]

\[ f_{\text{center,stopband}} = \frac{f_{c,LP} + f_{c,HP}}{2} \]
How a signal should look like after passing through a filter

Figure 6.17
Excitations and responses of lowpass and highpass CT filters.
**Practical Lowpass Filter**

**Basic Lowpass Filter**

\[ H(\omega) = \frac{1}{1 + j\omega RC} \]
\begin{align*}
\text{R} = 1,000 \text{ Ohm} & \quad \text{RC} = 1 \text{ msec} \\
\text{R} = 10,000 \text{ Ohm} & \quad \text{RC} = 10 \text{ msec} \\
\text{C} = 1 \text{ µF} & \\
\text{C} = 1 \text{ µF} & 
\end{align*}
**Single Stage Lowpass Filter**

\[ H(\omega) = \frac{1}{1 + j\omega RC} \]

**Two Stage Lowpass Filter**

\[ H(\omega) = \left(\frac{1}{1 + j\omega RC}\right)\left(\frac{1}{1 + j\omega RC}\right) \]
R = 1000 Ohm
C = 1 μF

Magnitude

Phase (rad)

Frequency in rads/sec

Single Stage

Dual Stage

R = 1000 Ohm
C = 1 μF
Practical Bandpass Filter

Basic Bandpass Filter

\[ H(\omega) = \frac{j\omega/RC}{(j\omega)^2 + j(\omega/RC) + (1/LC)} \]

\[ \omega_0 = \pm(1/\sqrt{LC}) \]
Figure 6.36
Magnitude and phase frequency responses of a practical RLC bandpass filter.
Discrete Time Filters

\[
x[n] \rightarrow \oplus \rightarrow y[n]
\]

\[
x[n] - \frac{1}{2} y[n-1] = x[n]
\]

**Impulse Response**

\[
h[n] = \left( \frac{1}{2} \right)^n u[n]
\]

**Transfer Function by taking DTFT of \( h[n] \)**

\[
H(F) = \frac{1}{1 - \frac{1}{2} e^{-2\pi F}}
\]
CT vs. DT Lowpass Filters

**Continuous Time**

\[ RC \frac{dy(t)}{dt} + y(t) = x(t) \]

\[ \Rightarrow RC \frac{dh(t)}{dt} + h(t) = \delta(t) \]

\[ \Rightarrow h(t) = e^{-t/RC} u(t) \]

**Discrete Time**

\[ y[n] - \frac{1}{2} y[n-1] = x[n] \]

\[ \Rightarrow h[n] - \frac{1}{2} h[n-1] = \delta[n] \]

\[ \Rightarrow h[n] = \left( \frac{1}{2} \right)^n u[n] \]
CT vs. DT Lowpass Filters … cont.

**Time Response**

\[ h(t) = e^{-t/RC} u(t) \]

\[ h[n] = \left(\frac{1}{2}\right)^n u[n] \]

**Frequency Response**

\[ |H(f)| \]

\[ |H(F)| \]
CT vs. DT Lowpass Filters … cont.

Magnitude Response

\[ |H(f)| \]

Phase Response

\[ \langle H(f) \rangle \]