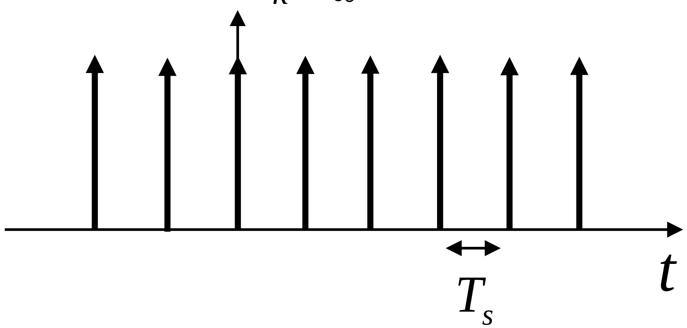
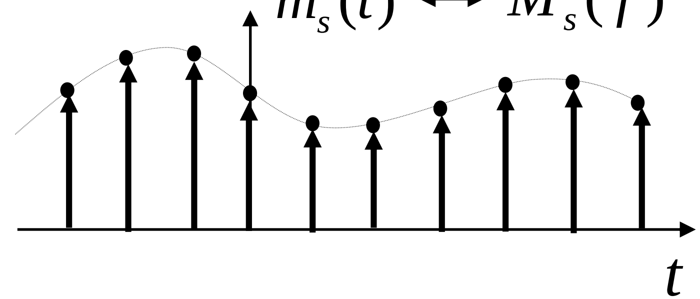
Application of Combined Fourier Series Transform (Sampling Theorem)

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \leftrightarrow X[k]$$

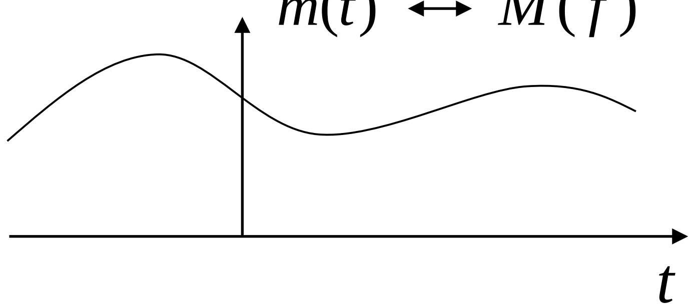


$$\otimes$$

$$m_s(t) \leftrightarrow M_s(f)$$



$$m(t) \leftrightarrow M(f)$$

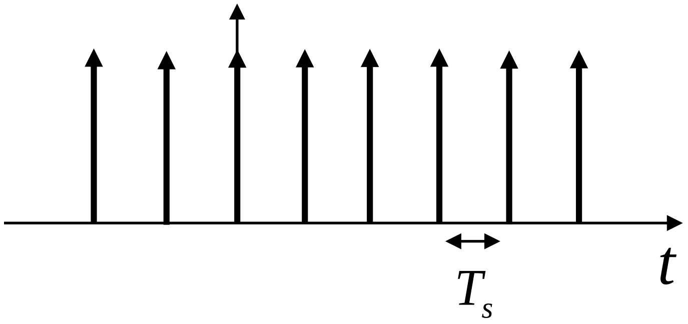


Sampling Frequency

$$f_s = \frac{1}{T_s}$$

$$x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT_s)\longleftrightarrow\ X[k]$$

We need Fourier Series



$$x(t)=X[0]+\sum_{k=1}^{k=\infty}(X_c[k]\cos(2\pi kf_st)+X_s[k]\sin(2\pi kf_st))$$

$$X[0]=\frac{1}{T_s}\int_{t_0}^{t_0+T_s}x(t)dt=\frac{1}{T_s}\int_{0}^{T_s}\delta(t)dt=\frac{1}{T_s}$$

$$X_s[k]=\frac{2}{T_s}\int_{t_0}^{t_0+T_s}x(t)\sin(2\pi kf_st)dt=0$$

$$X_c[k] = \frac{2}{T_s} \int_{t_0}^{t_0+T_s} x(t)\cos(2\pi k f_s t)dt$$

$$= \frac{2}{T_s} \int_{0}^{T_s} \delta(t)\cos(2\pi k f_s t)dt = \frac{2}{T_s}\cos(0) = \frac{2}{T_s}$$

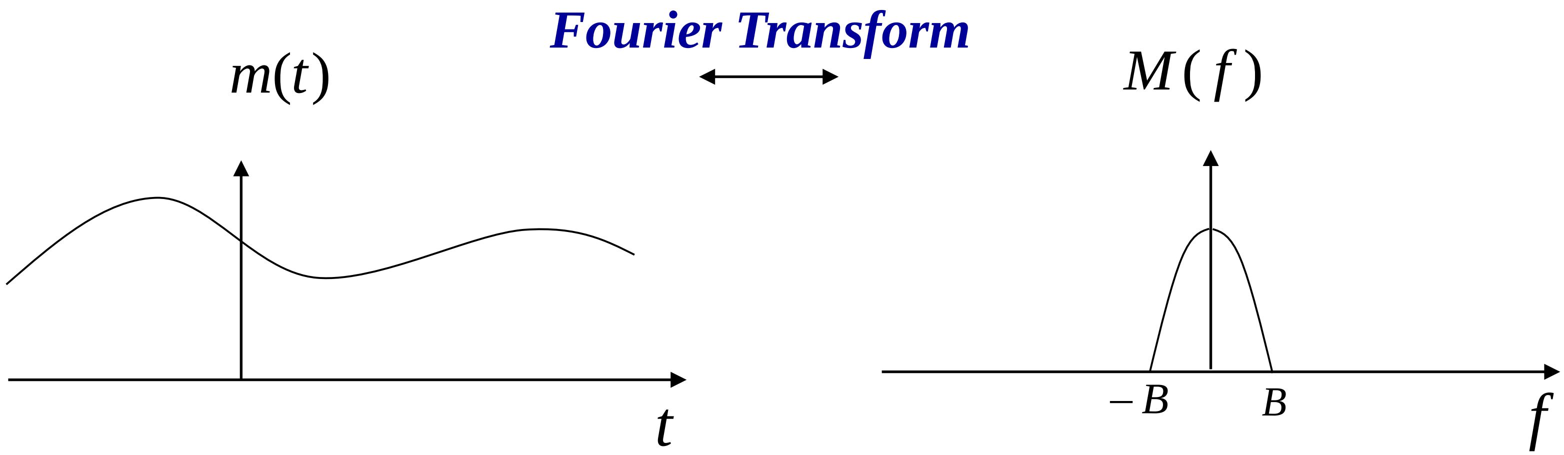
Let's go back to Fourier Series representation

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} (X_c[k]\cos(2\pi k f_s t) + X_s[k]\sin(2\pi k f_s t))$$

$$= X[0] + \sum_{k=1}^{k=\infty} X_c[k]\cos(2\pi k f_s t)$$

$$\Rightarrow x(t) = \frac{1}{T_s} + \frac{2}{T_s}\sum_{k=1}^{k=\infty}\cos(2\pi k f_s t)$$

What about m(t)? – need Fourier Transform



B is the Bandwidth of m(t)

We now have x(t) as frequency representation via Fourier Series

$$x(t) = \frac{1}{T_s} + \frac{2}{T_s}\sum_{k=1}^{k=\infty}\cos(2\pi k f_s t)$$

And m(t) as Frequency representation via Fourier Transform

$$m(t) \leftrightarrow M(f)$$

Let's multiply m(t) with x(t)

$$m(t)x(t) = \frac{1}{T_s}m(t) + \frac{2}{T_s}\sum_{k=1}^{k=\infty} m(t)\cos(2\pi k f_s t)$$

What is the frequency representation of m(t)x(t)?

$$m(t)x(t) = \frac{1}{T_s} m(t) + \frac{2}{T_s} \sum_{k=1}^{k=\infty} m(t)\cos(2\pi k f_s t)$$

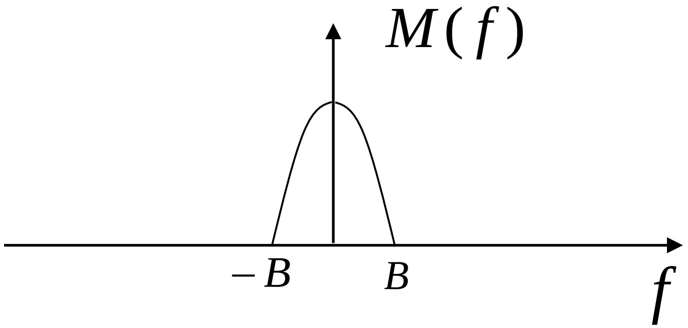
We need to take Fourier Transform

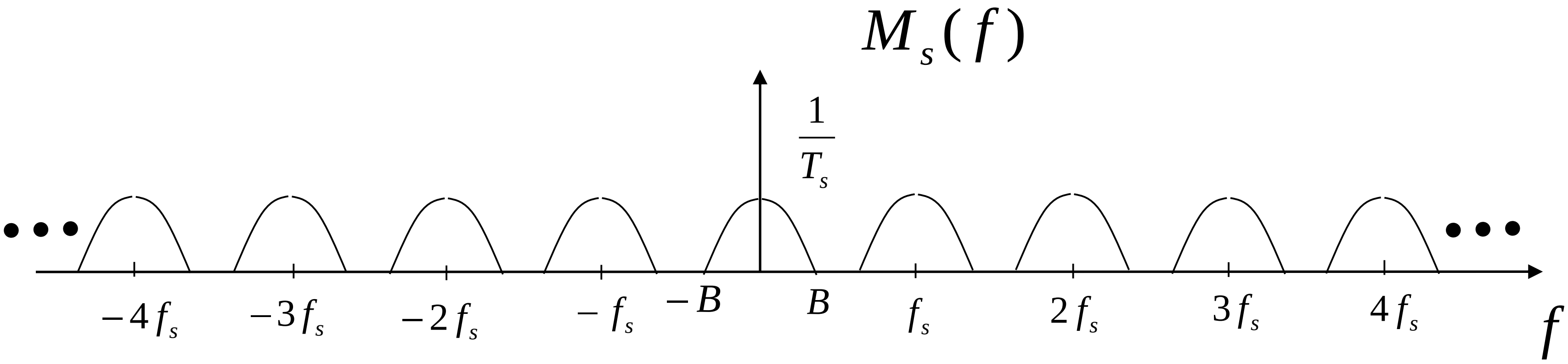
$$F[m(t)x(t)] = F[\frac{1}{T_s} m(t) + \frac{2}{T_s} \sum_{k=1}^{k=\infty} m(t)\cos(2\pi k f_s t)]$$

$$M_s(f) = \frac{1}{T_s} M(f) + \frac{2}{T_s} \sum_{k=1}^{k=\infty} [\frac{1}{2}(M(f - kf_s) + M(f + kf_s))]$$

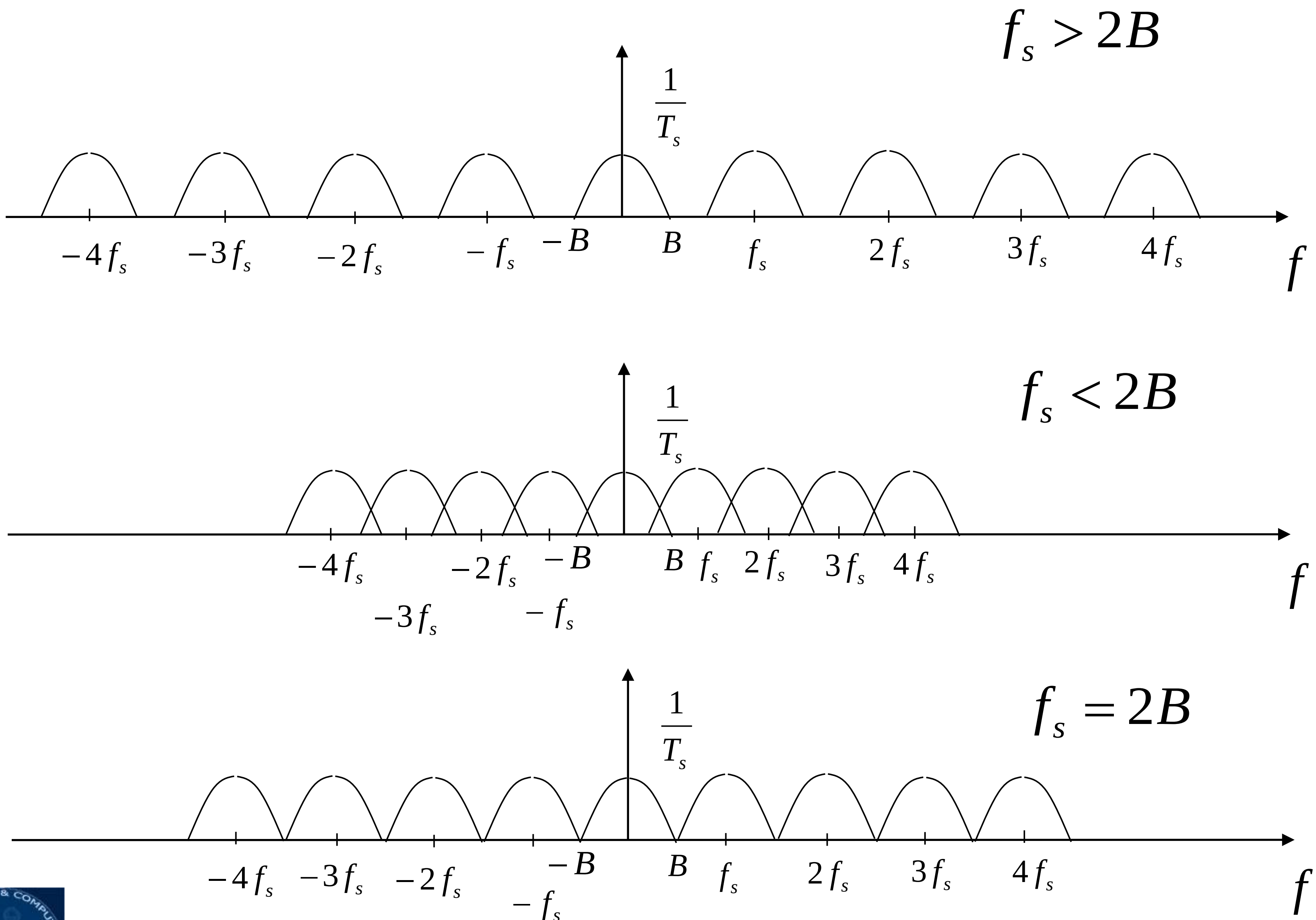
$$= \frac{1}{T_s} M(f) + \frac{1}{T_s} \sum_{k=1}^{k=\infty} [M(f - kf_s) + M(f + kf_s)]$$

$$F[m(t)x(t)] = M_s(f) = \frac{1}{T_s}M(f) + \frac{1}{T_s}\sum_{k=1}^{k=\infty}[M(f-kf_s)+M(f+kf_s)]$$





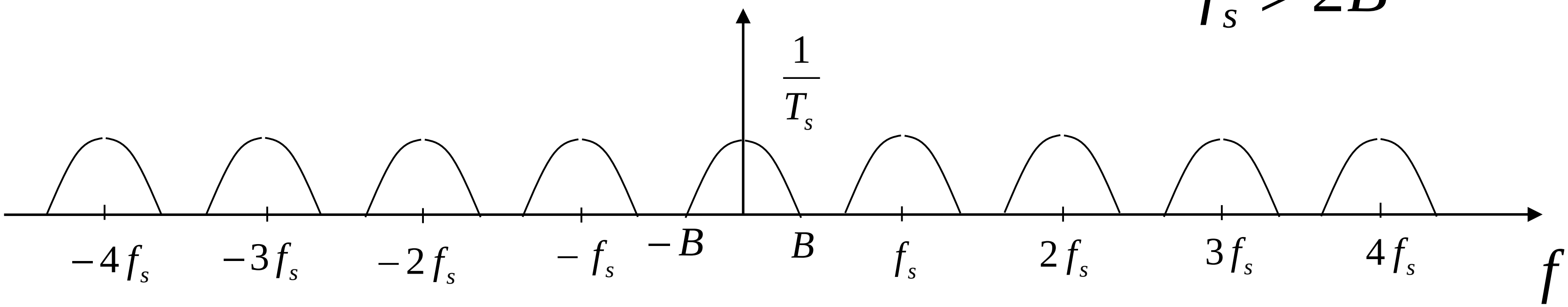
$M_s(f)$



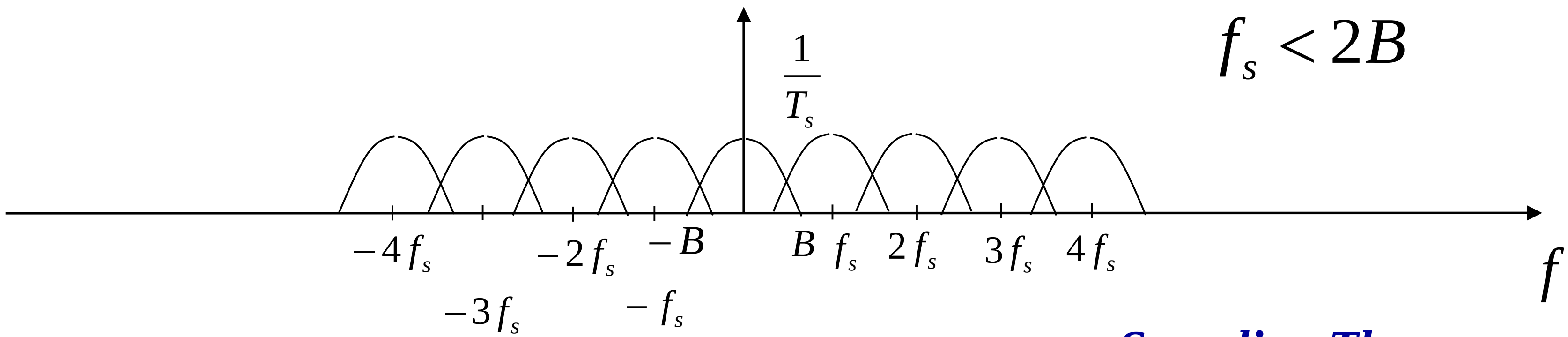
$f_s > 2B$
$\frac{1}{T_s}$
$-4f_s$ $-3f_s$ $-2f_s$ $-f_s$ $-B$ B f_s $2f_s$ $3f_s$ $4f_s$
f
$f_s < 2B$
$\frac{1}{T_s}$
$-4f_s$ $-2f_s$ $-B$ B f_s $2f_s$ $3f_s$ $4f_s$
$-3f_s$ $-f_s$
f
$f_s = 2B$
$\frac{1}{T_s}$
$-4f_s$ $-3f_s$ $-2f_s$ $-B$ B f_s $2f_s$ $3f_s$ $4f_s$
$-f_s$
f

$$M_s(f)$$

$f_s > 2B$

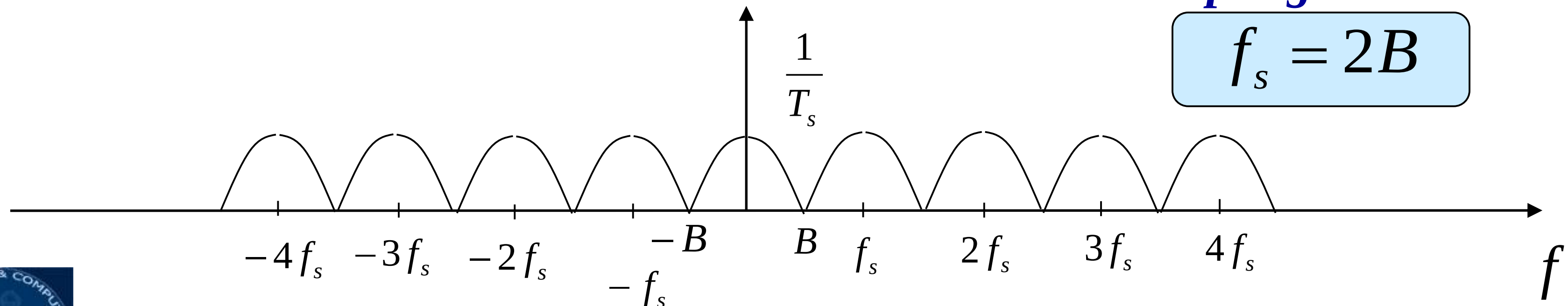


$f_s < 2B$



Sampling Theorem

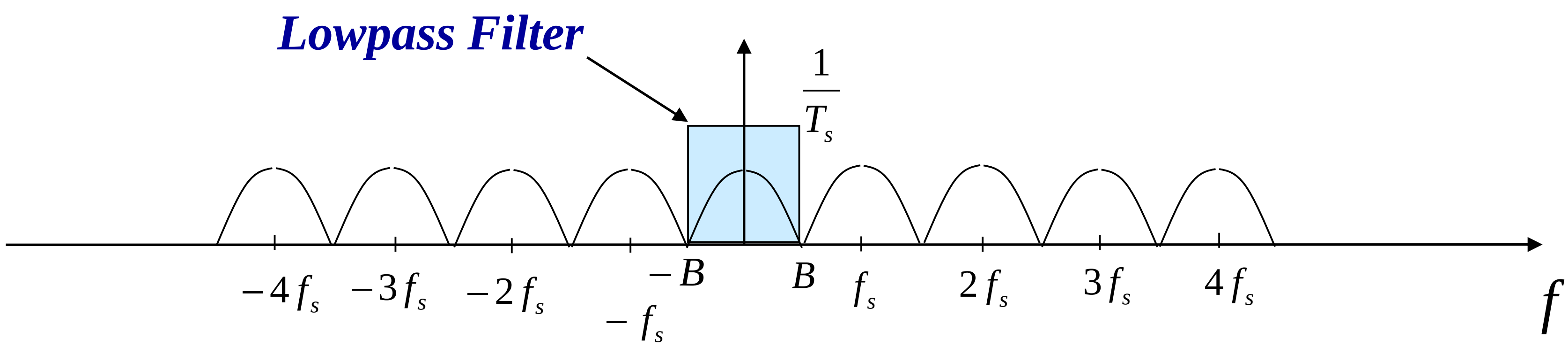
$$f_s = 2B$$



Sampling Theorem

Sampling frequency should be at least equal to or greater than twice the bandwidth of the message signal for successful recovery of the signal from it's samples

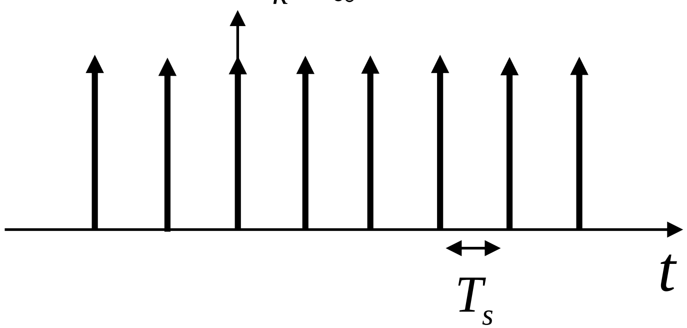
$$f_s \geq 2B$$



bandwidth of lowpass filter = B Hz

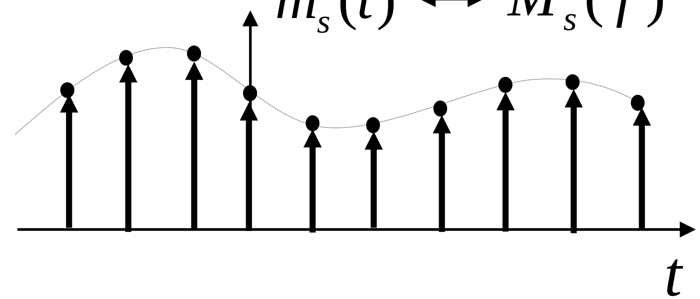
A practical problem with Sampling

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \leftrightarrow X[k]$$

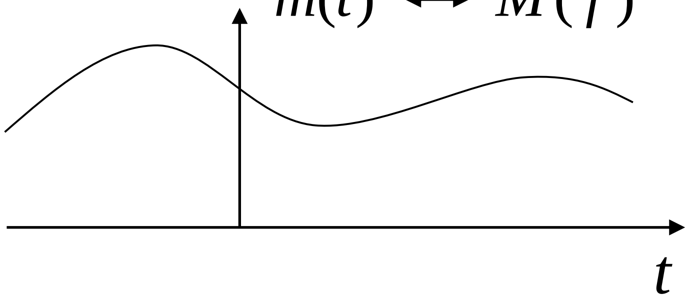


Not practical to generate!

$$m_s(t) \leftrightarrow M_s(f)$$



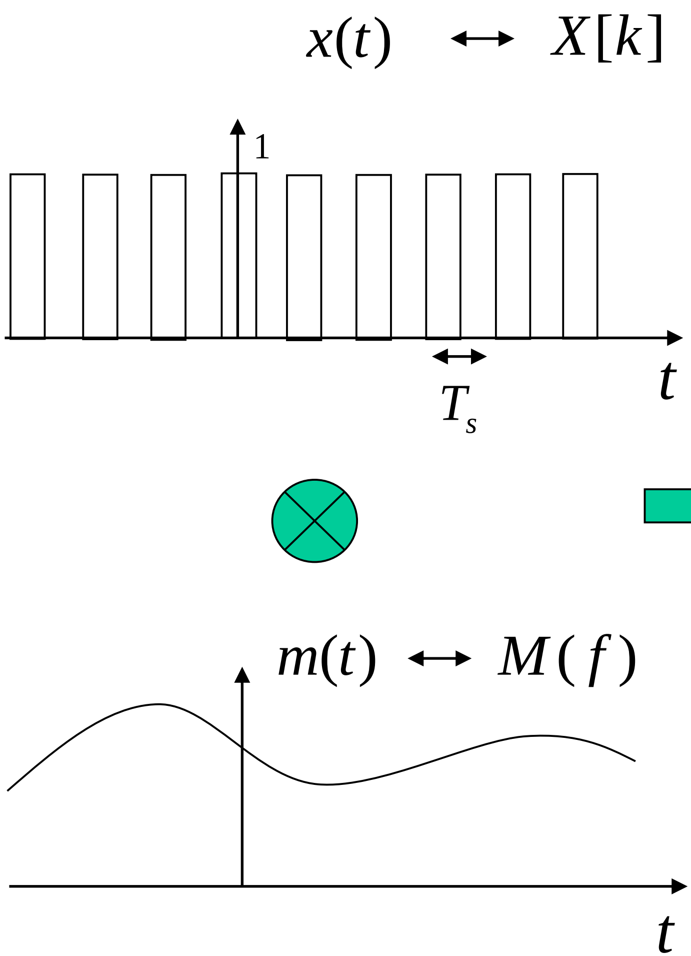
$$m(t) \leftrightarrow M(f)$$



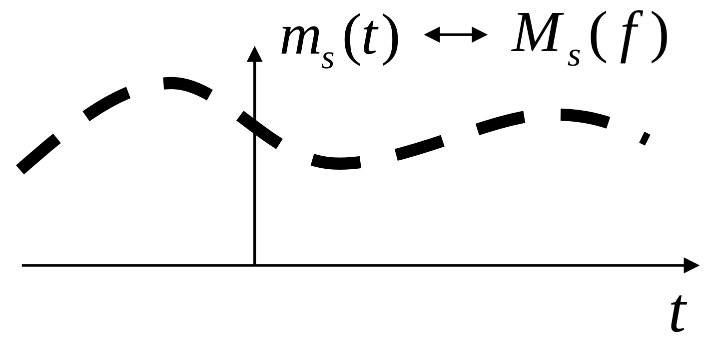
Sampling Frequency

$$f_s = \frac{1}{T_s}$$

Let's Try Square Wave



Problem Resolved

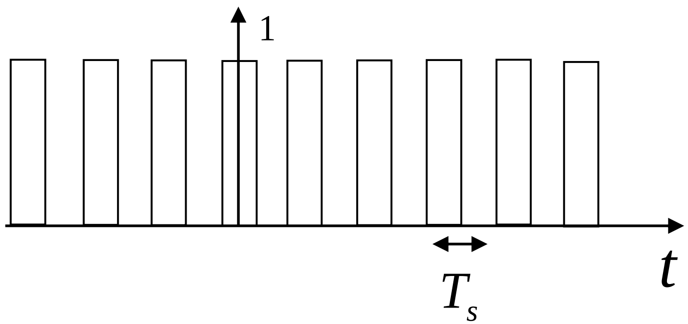


Sampling Frequency

$$f_s = \frac{1}{T_s}$$

$x(t) \quad \longleftrightarrow \quad X[k]$

We need Fourier Series



We know

$$X[0]=\frac{1}{2} \qquad X_c[k]=\frac{4}{\pi k}\sin(\frac{\pi k}{2}) \qquad X_s[k]=0$$

and

$$x(t)=X[0]+\sum_{k=1}^{k=\infty}(X_c[k]\cos(2\pi k f_s t)+X_s[k]\sin(2\pi k f_s t))$$

$$\Rightarrow x(t)=\frac{1}{2}+\sum_{k=1}^{k=\infty}\frac{4}{\pi k}\sin(\frac{\pi k}{2})\cos(2\pi k f_s t)$$

We now have x(t) as frequency representation via Fourier Series

$$x(t) = \frac{1}{2} + \sum_{k=1}^{k=\infty} \frac{4}{\pi k} \sin(\frac{\pi k}{2}) \cos(2\pi k f_s t)$$

And m(t) as Frequency representation via Fourier Transform

$$m(t) \leftrightarrow M(f)$$

Let's multiply m(t) with x(t)

$$m(t)x(t) = \frac{1}{2} m(t) + \sum_{k=1}^{k=\infty} \frac{4}{\pi k} \sin(\frac{\pi k}{2}) m(t) \cos(2\pi k f_s t)$$

What is the frequency representation of m(t)x(t)?

$$m(t)x(t) = \frac{1}{2}m(t) + \sum_{k=1}^{k=\infty} \frac{4}{\pi k}\sin(\frac{\pi k}{2})m(t)\cos(2\pi k f_s t)$$

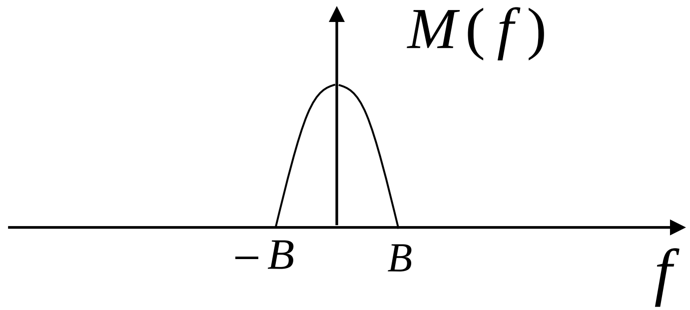
We need to take Fourier Transform

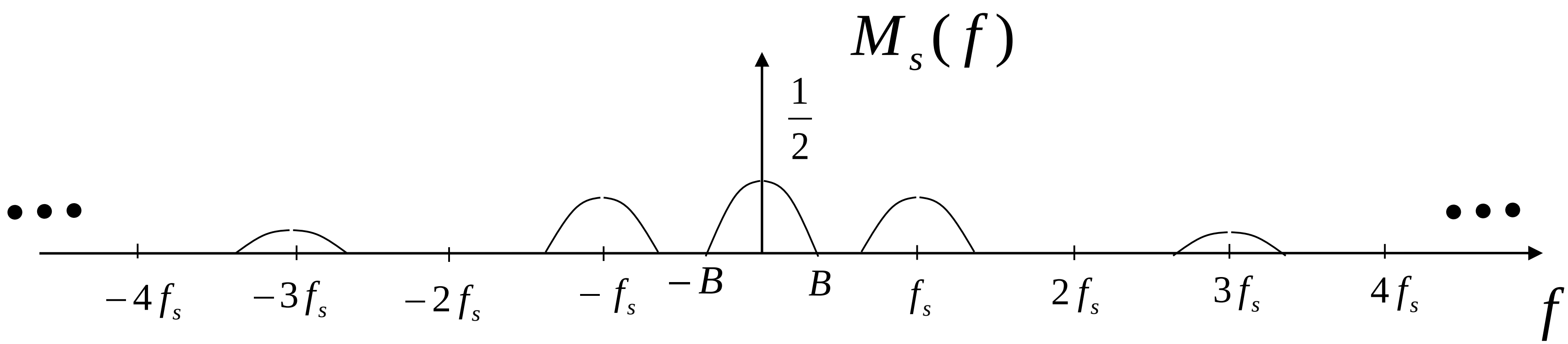
$$F[m(t)x(t)] = F[\frac{1}{2}m(t) + \sum_{k=1}^{k=\infty} \frac{4}{\pi k}\sin(\frac{\pi k}{2})m(t)\cos(2\pi k f_s t)]$$

$$M_s(f) = \frac{1}{2}M(f) + \sum_{k=1}^{k=\infty} \frac{4}{\pi k}\sin(\frac{\pi k}{2})[\frac{1}{2}(M(f - kf_s) + M(f + kf_s))]$$

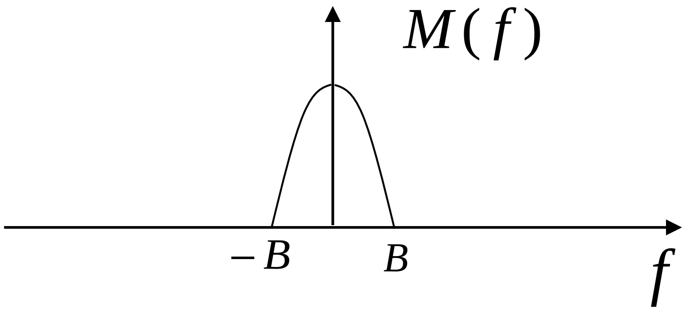
$$= \frac{1}{2}M(f) + \sum_{k=1}^{k=\infty} \frac{2}{\pi k}\sin(\frac{\pi k}{2})[(M(f - kf_s) + M(f + kf_s))]$$

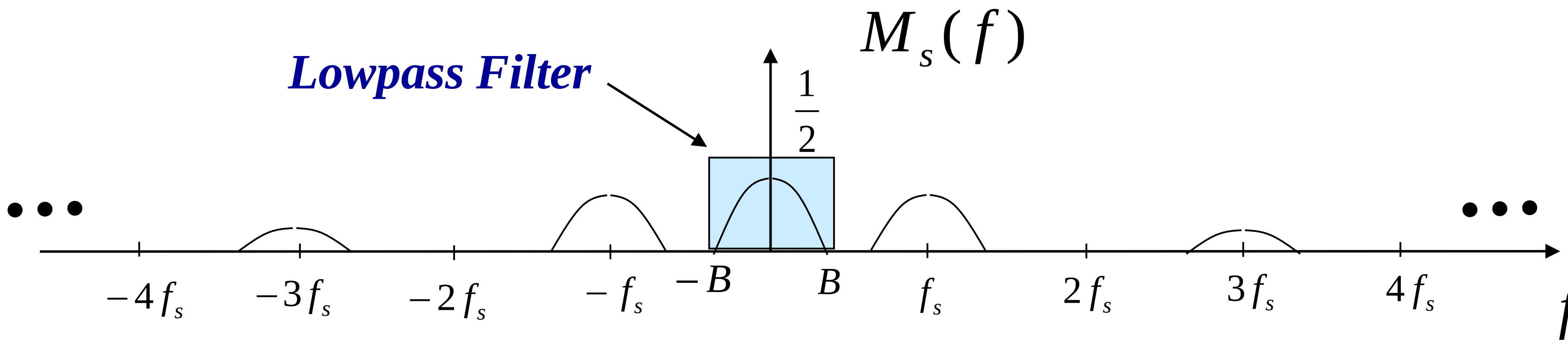
$$F[m(t)x(t)] = M_s(f) = \frac{1}{2}M(f) + \sum_{k=1}^{k=\infty} \frac{4}{\pi k}\sin([M(f-kf_s) + M(f+kf_s)]$$





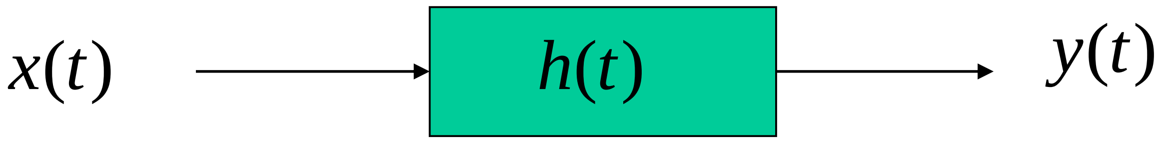
$$F[m(t)x(t)] = M_s(f) = \frac{1}{2}M(f) + \sum_{k=1}^{k=\infty}\frac{4}{\pi k}\sin([M(f-kf_s)+M(f+kf_s)]$$





Sampling Theorem still works with practical samples

Distortion and Linear systems



$h(t)$ - ***Impulse Response of a Linear Time Invariant System***

$$y(t) = x(t) * h(t)$$

$$\Rightarrow \quad Y(\omega) = X(\omega)H(\omega) \qquad\qquad h(t) \Leftrightarrow H(\omega)$$

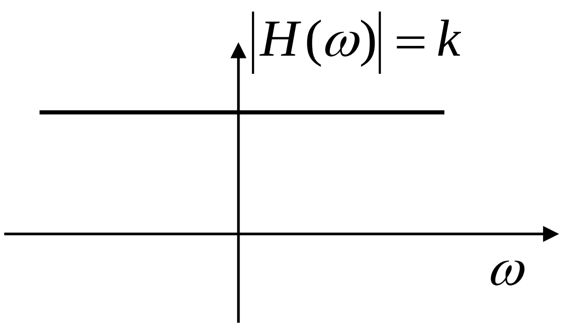
Distortionless System

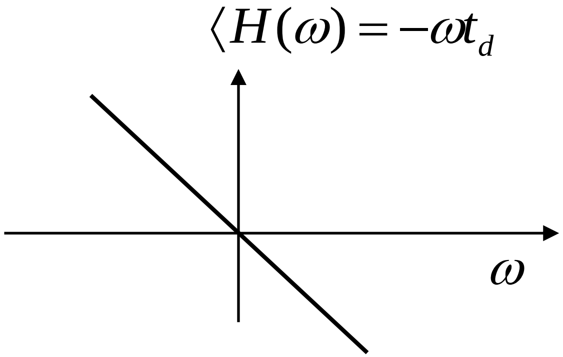
$$y(t) = kx(t - t_d)$$

$$\Rightarrow \quad Y(\omega) = kX(\omega)e^{-j\omega t_d}$$

we know $$Y(\omega) = X(\omega)H(\omega)$$

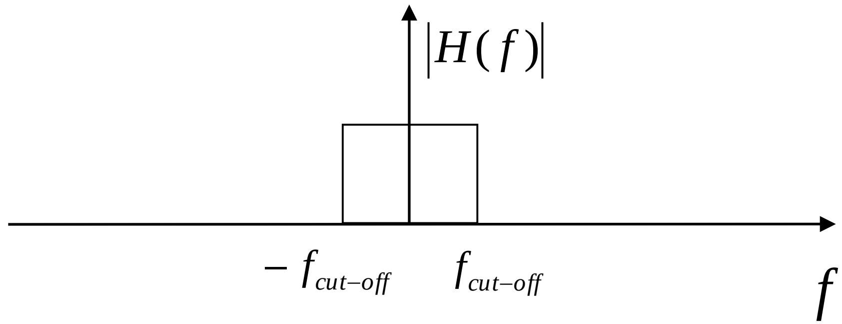
$$\Rightarrow \quad H(\omega) = ke^{-j\omega t_d}$$





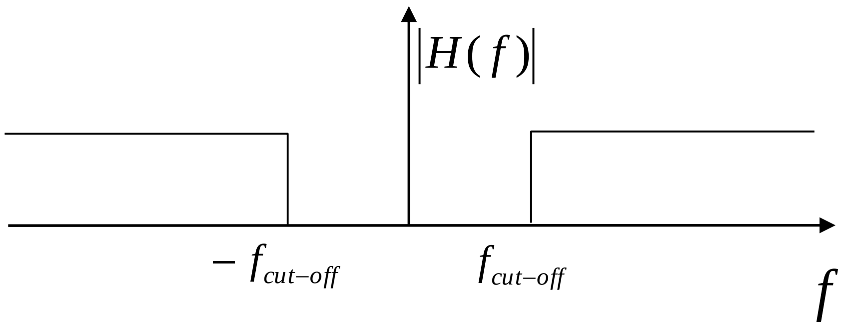
Filters

Lowpass Filter



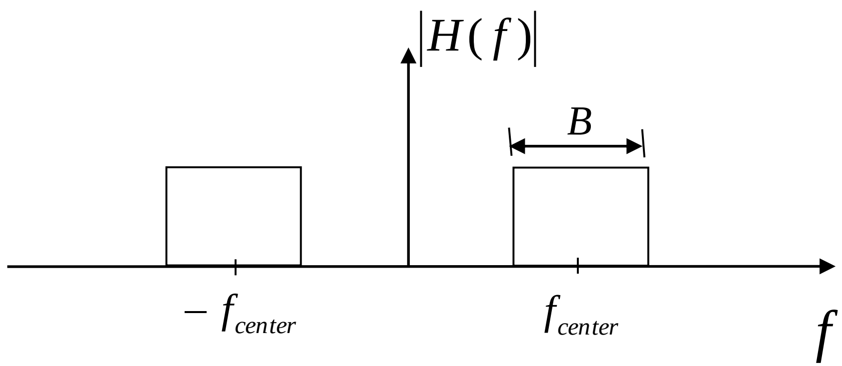
Cut-off Frequency

Highpass Filter



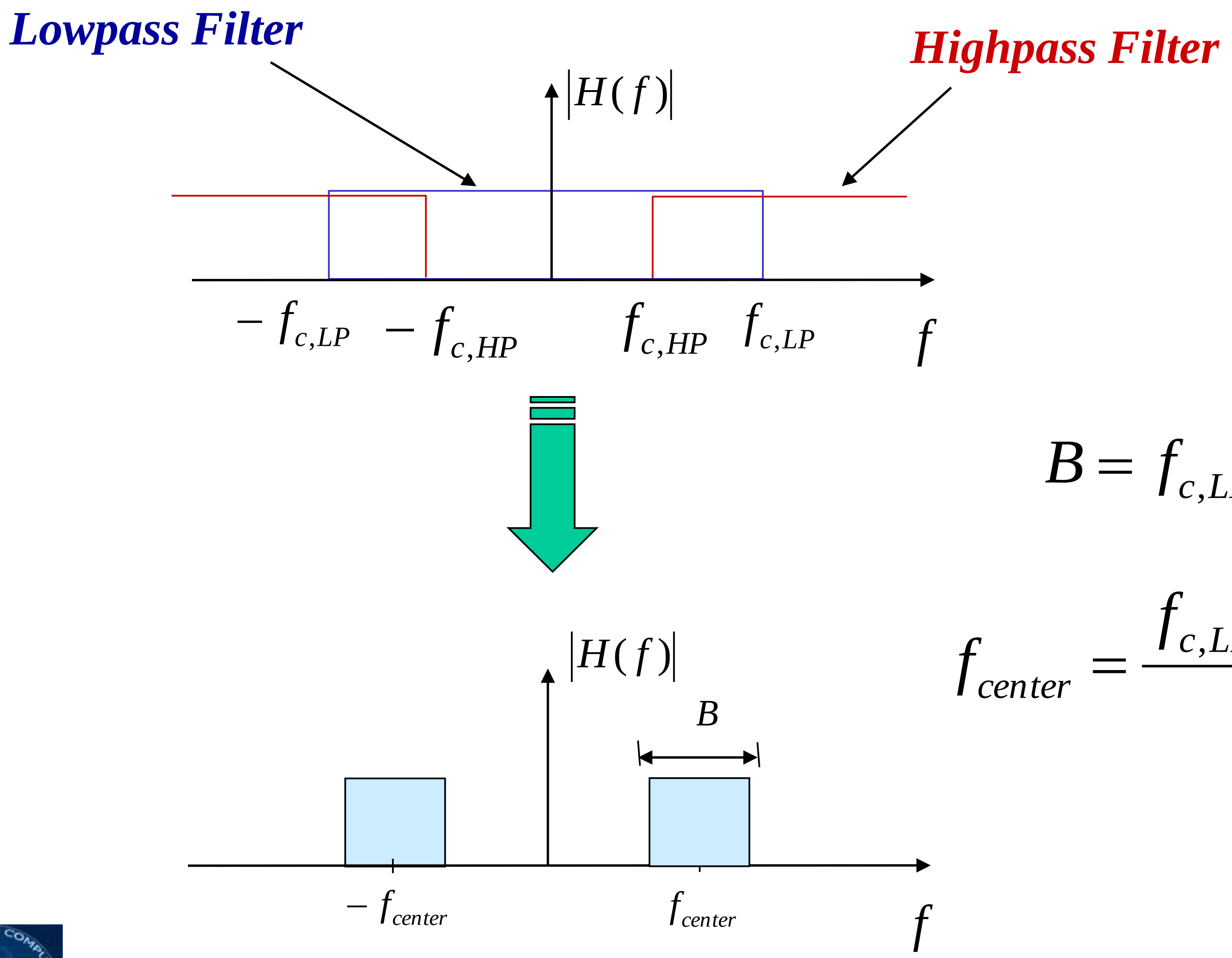
Cut-off Frequency

Bandpass Filter



Center Frequency
Bandwidth

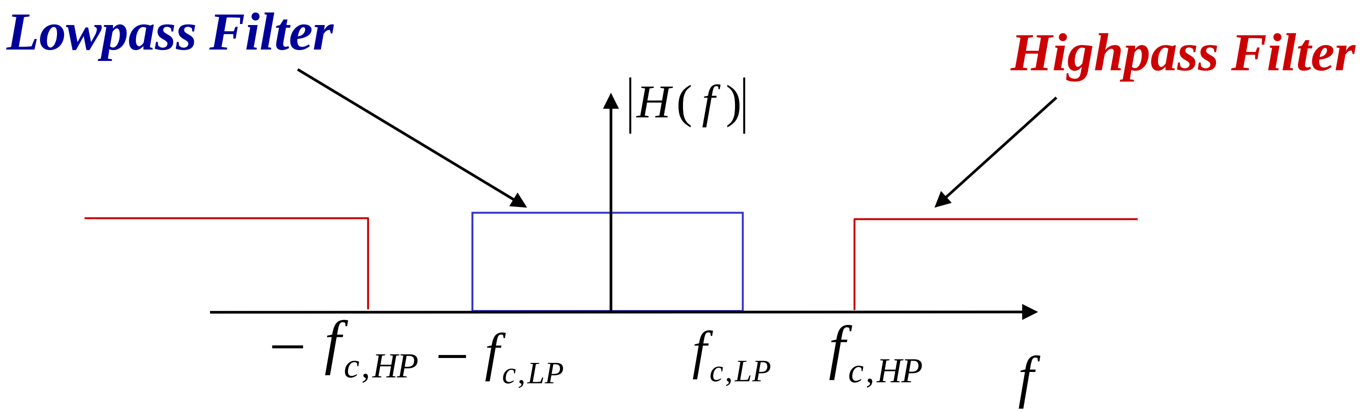
Lowpass and Highpass combined to generate Bandpass Filter



$$B = f_{c,LP} - f_{c,HP}$$

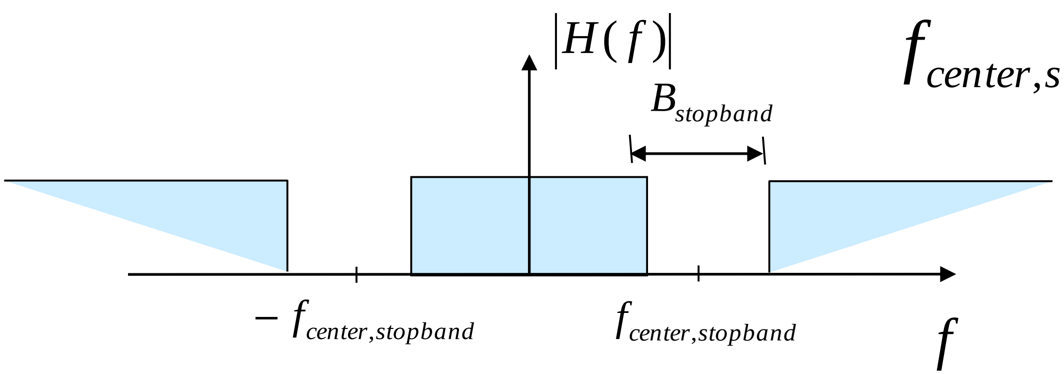
$$f_{center} = \frac{f_{c,LP} + f_{c,HP}}{2}$$

Lowpass and Highpass combined to generate Bandstop Filter



$$B_{stopband} = f_{c,HP} - f_{c,LP}$$

$$f_{center,stopband} = \frac{f_{c,LP} + f_{c,HP}}{2}$$



How a signal should look like after passing through a filter

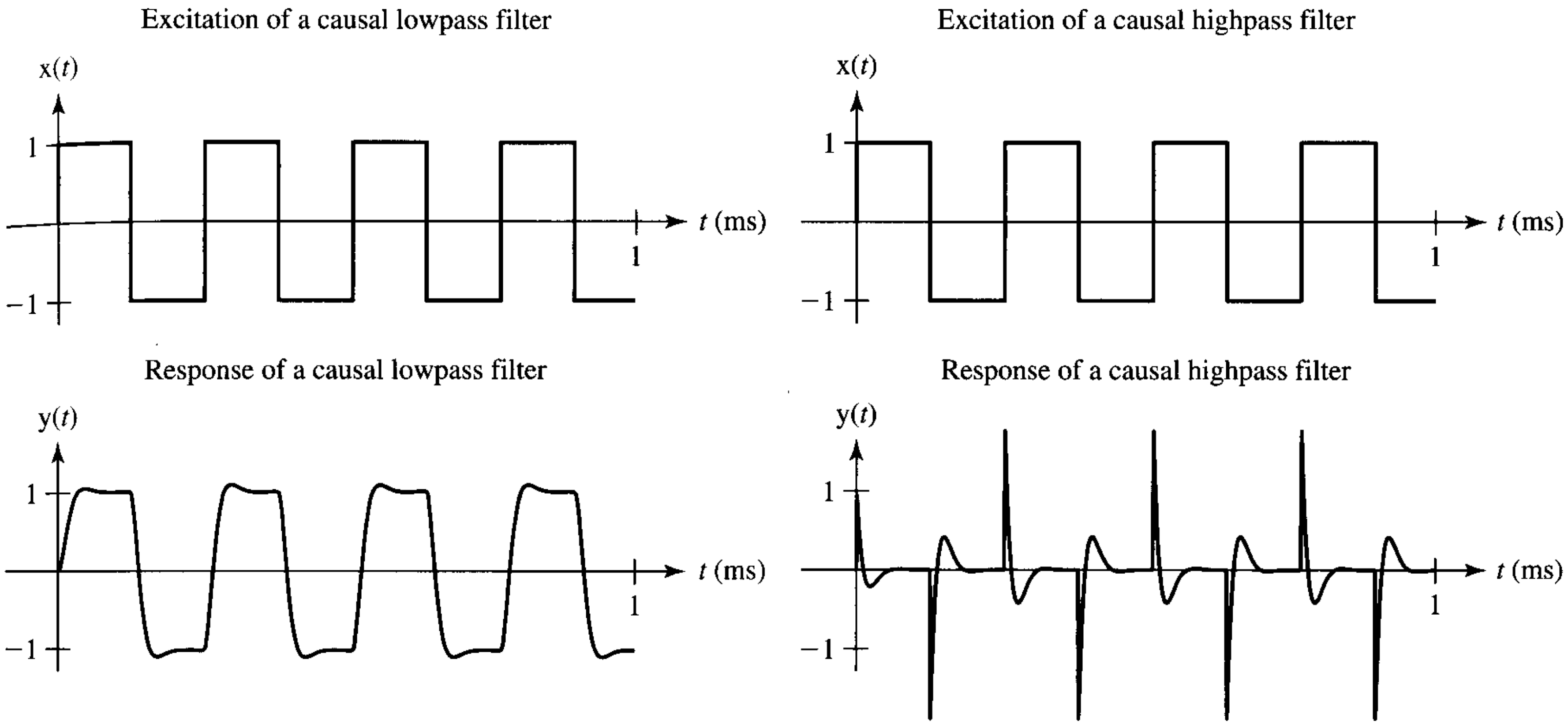
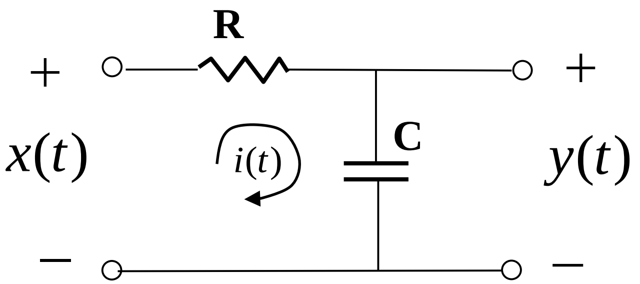


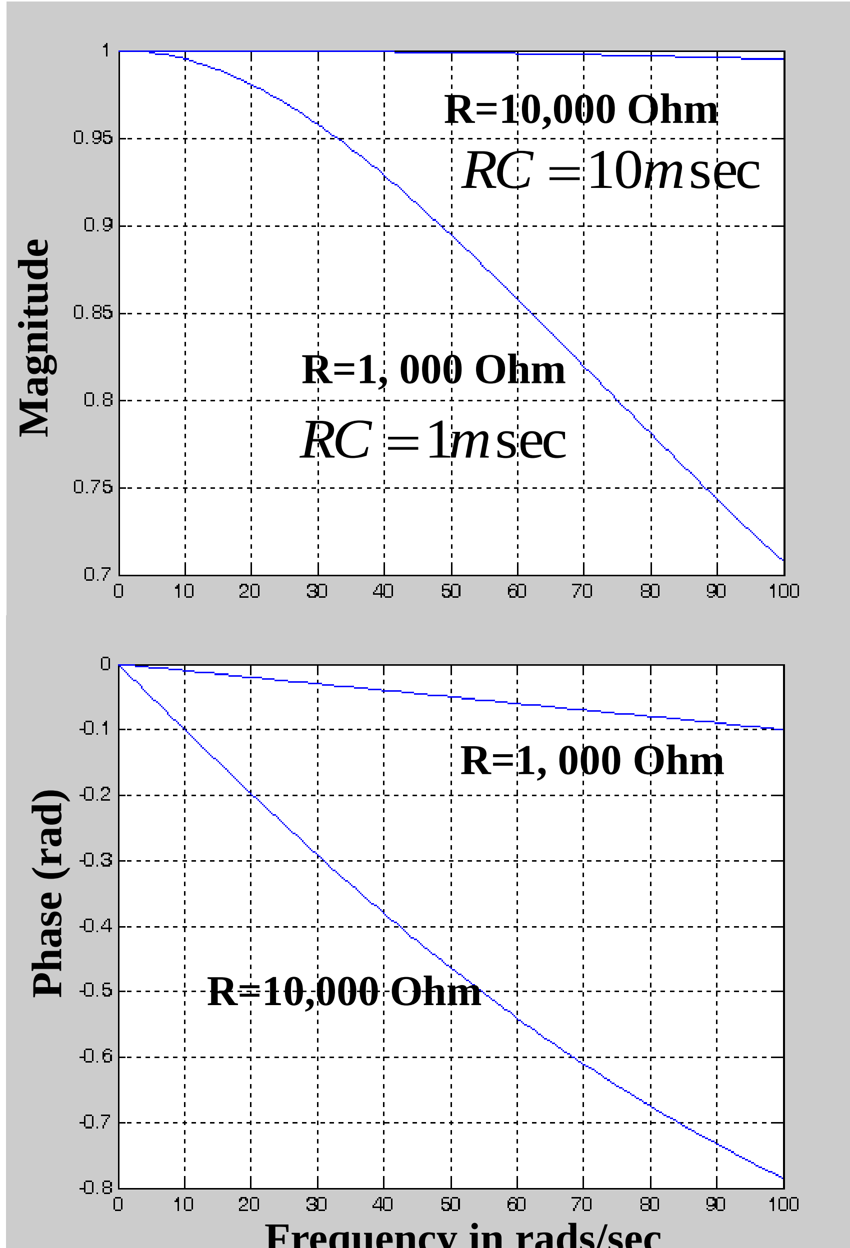
Figure 6.17
Excitations and responses of lowpass and highpass CT filters.

Practical Lowpass Filter

Basic Lowpass Filter



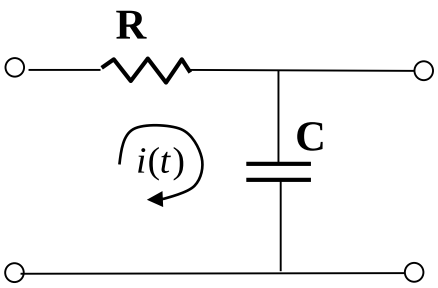
$$H(\omega) = \frac{1}{1 + j\omega RC}$$


Magnitude
1
0.95
0.9
0.85
0.8
0.75
0.7
0 10 20 30 40 50 60 70 80 90 100
R=10,000 Ohm
RC = 10 msec
R=1, 000 Ohm
RC = 1 msec
Phase (rad)
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
0 10 20 30 40 50 60 70 80 90 100
R=1, 000 Ohm
R=10,000 Ohm
Frequency in rads/sec

C=1 μF

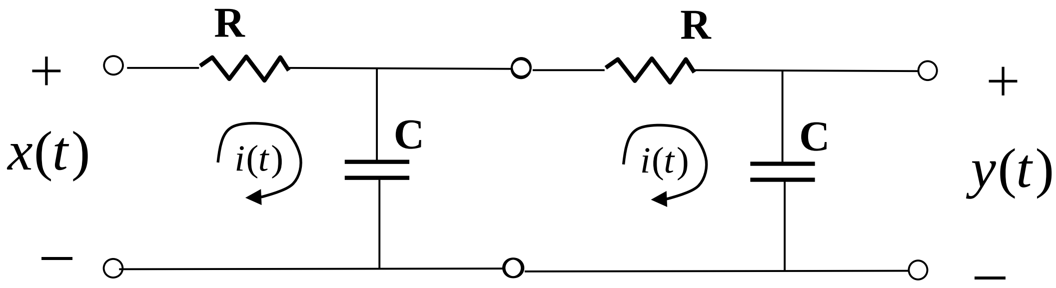
C=1 μF

Single Stage Lowpass Filter

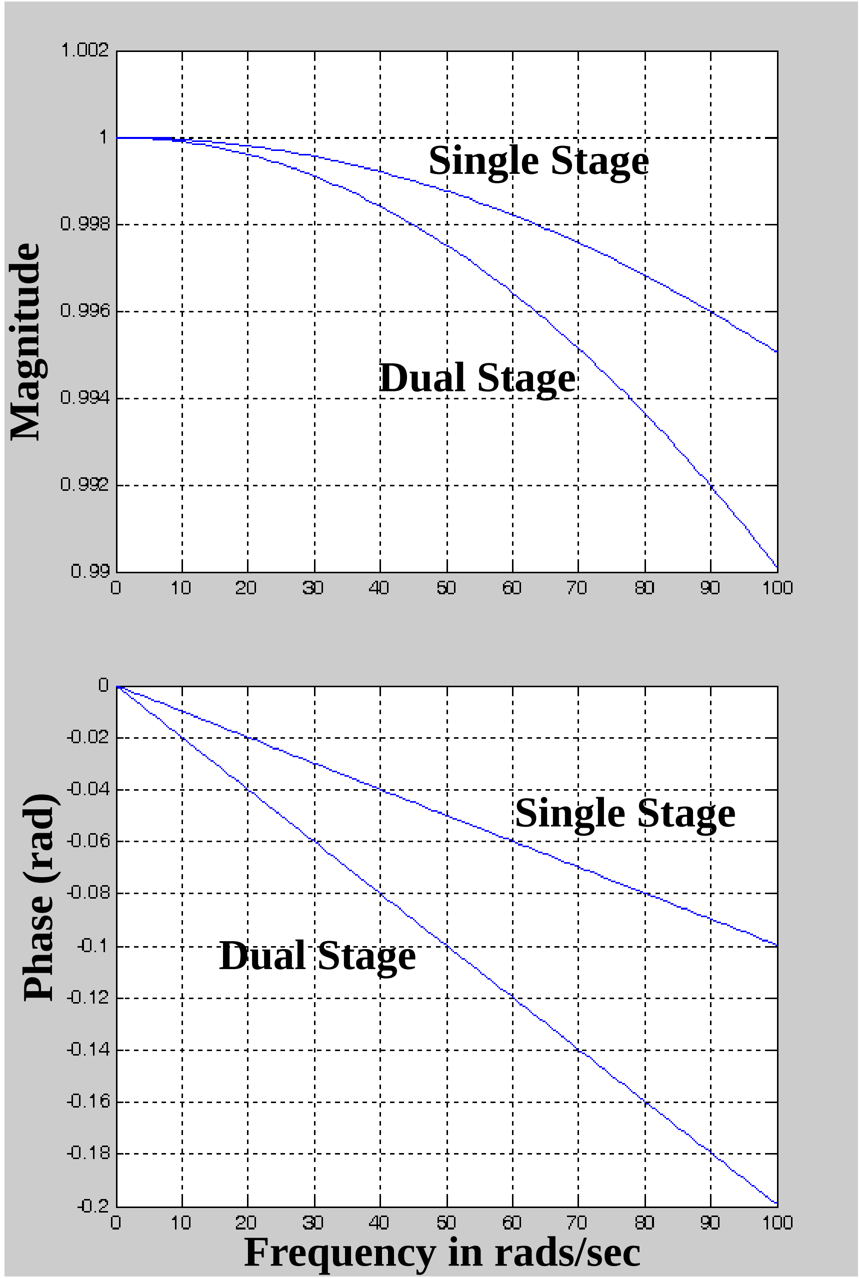


$$H(\omega) = \frac{1}{1 + j\omega RC}$$

Two Stage Lowpass Filter



$$H(\omega) = \left(\frac{1}{1 + j\omega RC}\right)\left(\frac{1}{1 + j\omega RC}\right)$$

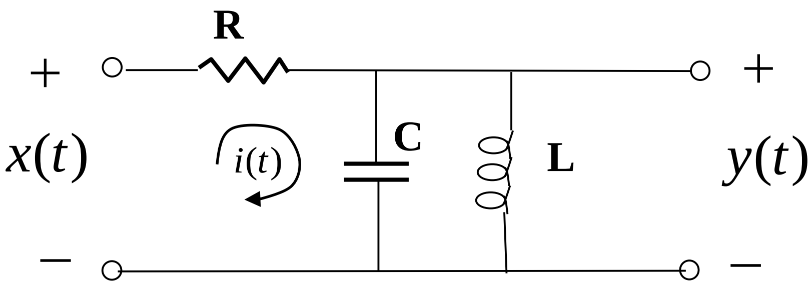


R=1000 Ohm
C=1 μF

R=1000 Ohm
C=1 μF

Practical Bandpass Filter

Basic Bandpass Filter



$$H(\omega) = \frac{j\omega / RC}{(j\omega)^2 + j(\omega / RC) + (1/LC)}$$

$$\omega_0 = \pm(1/\sqrt{LC})$$

Magnitude and Phase Response of BPF

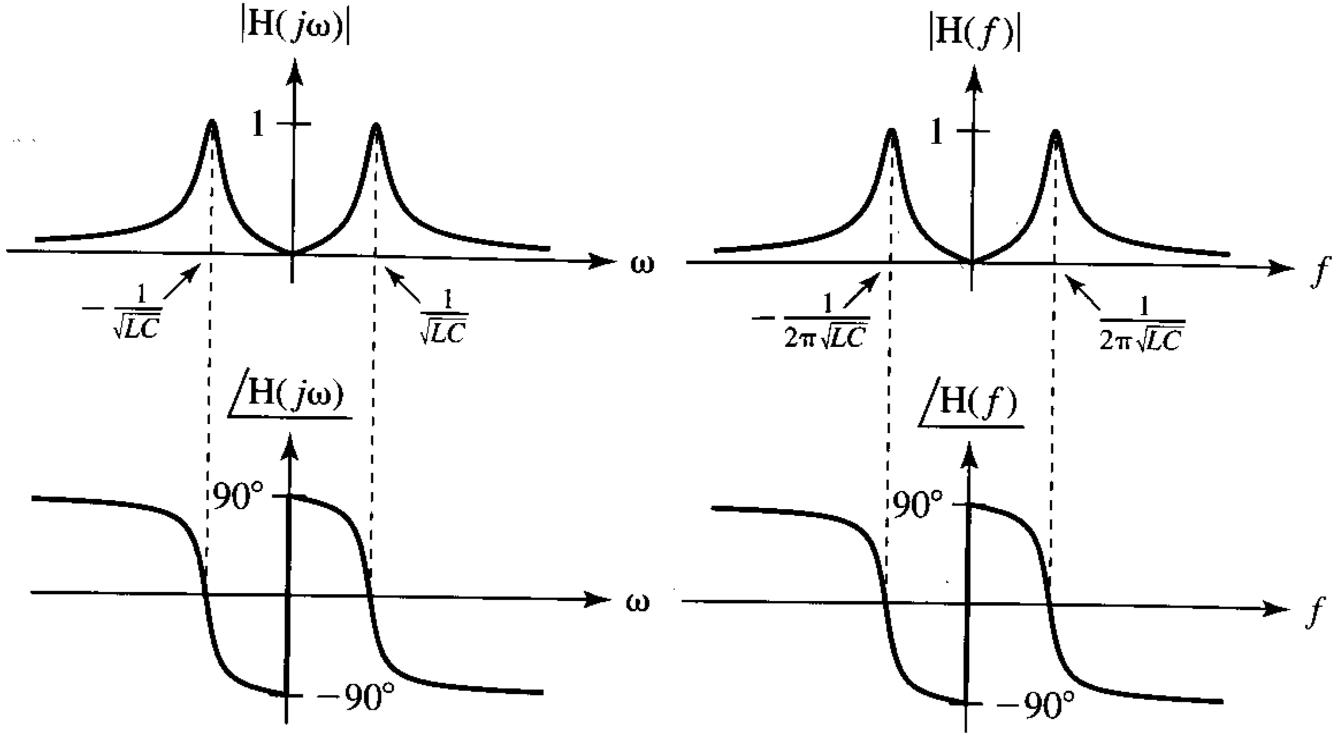
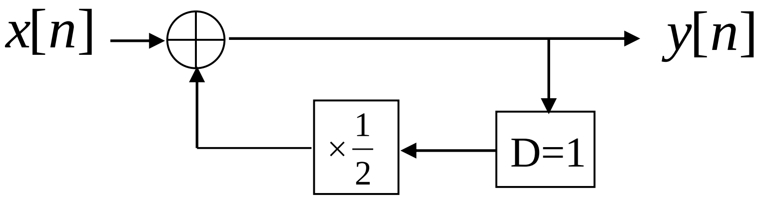


Figure 6.36
Magnitude and phase frequency responses of a practical *RLC* bandpass filter.

Discrete Time Filters



$$y[n]-\frac{1}{2}y[n-1]=x[n]$$

Impulse Response

$$h[n]=\left(\frac{1}{2}\right)^{n}u[n]$$

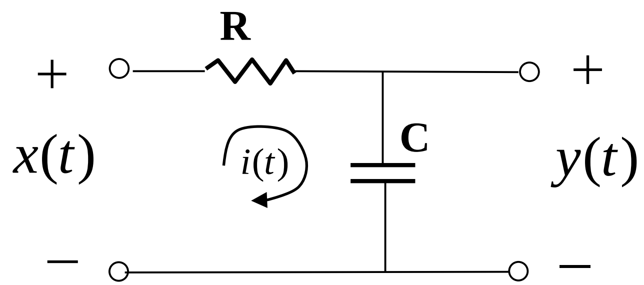
Transfer Function by taking DTFT of h[n]

$$H(F)=\frac{1}{1-\frac{1}{2}e^{-2\pi F}}$$

CT vs. DT Lowpass Filters

Reminder Slide

Continuous Time

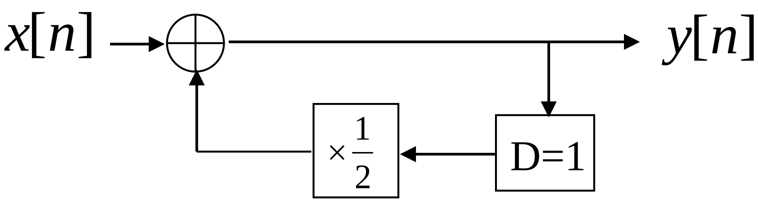


$$RC\frac{dy(t)}{dt}+y(t)=x(t)$$

$$\Rightarrow RC\frac{dh(t)}{dt}+h(t)=\delta(t)$$

$$\Rightarrow h(t)=e^{-t/RC}u(t)$$

Discrete Time



$$y[n]-\frac{1}{2}y[n-1]=x[n]$$

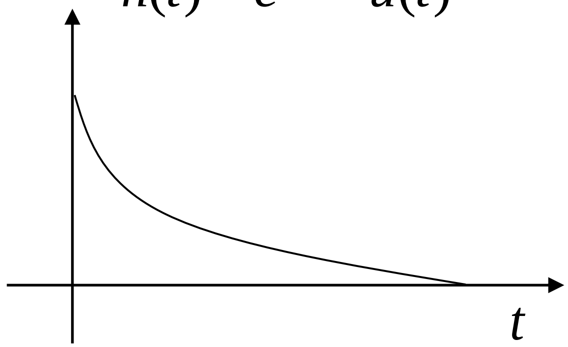
$$\Rightarrow h[n]-\frac{1}{2}h[n-1]=\delta[n]$$

$$\Rightarrow h[n]=\left(\frac{1}{2}\right)^n u[n]$$

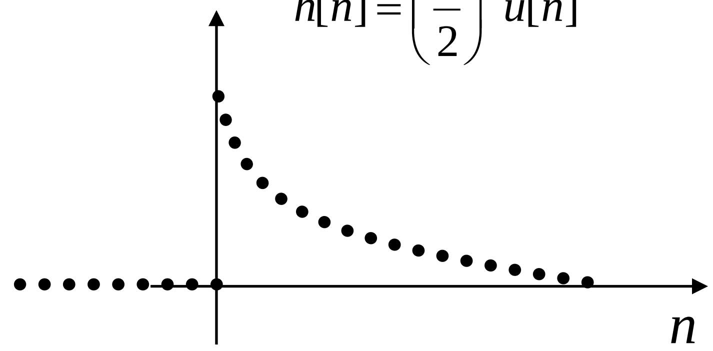
CT vs. DT Lowpass Filters … cont.

Time Response

$$h(t) = e^{-t/RC}u(t)$$

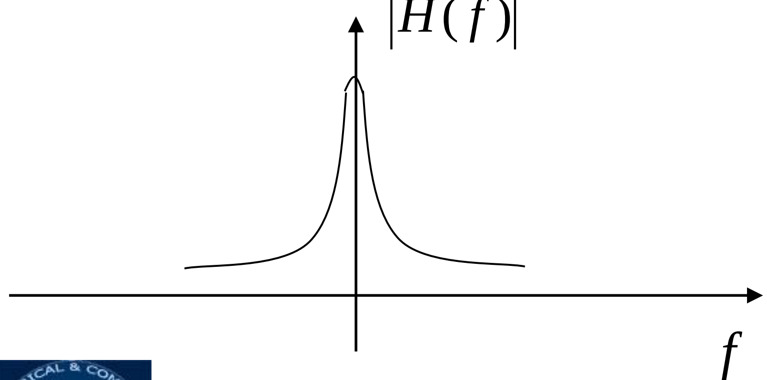


$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

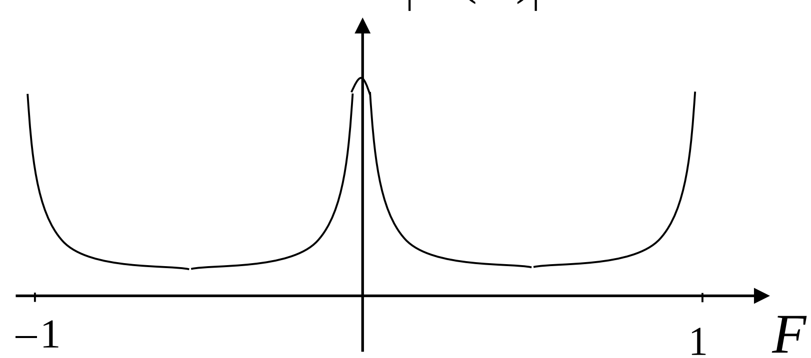


Frequency Response

$$|H(f)|$$

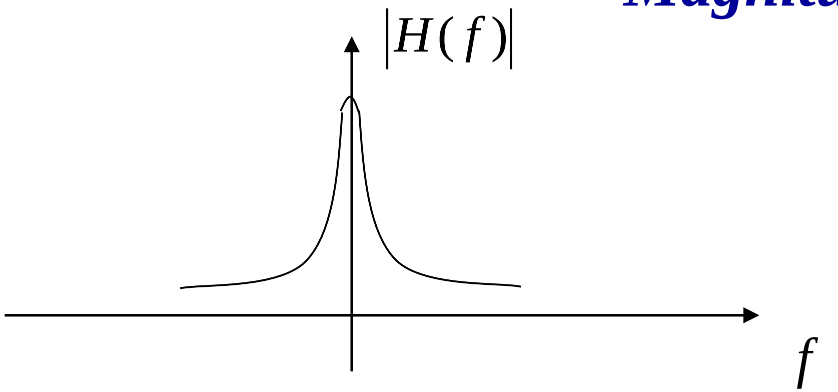


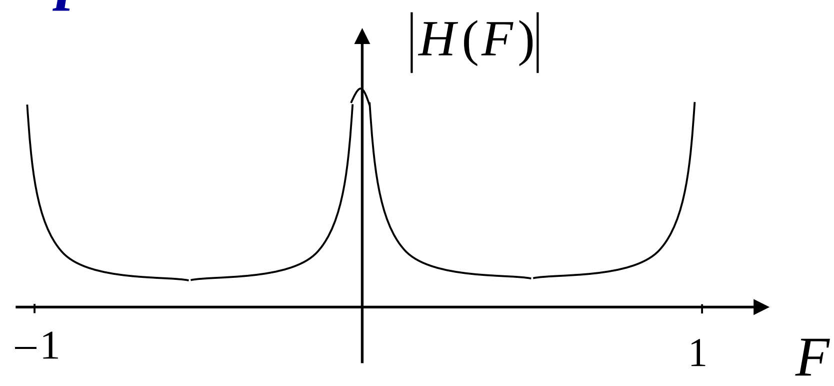
$$|H(F)|$$



CT vs. DT Lowpass Filters … cont.

Magnitude Response





Phase Response