Concept of probability

- ***Random Experiment***
- ***Number of Outcomes - Events***
- ***Sample Space – A correction of all possible events***
- ***Complement***
- ***Union***
- ***Intersection***

Relative Frequency and Probability

$$f(A)=\lim_{N\to\infty}\frac{N(A)}{N}$$

$$P(A)=\lim_{N\to\infty}\frac{N(A)}{N}$$

Important Results

$P(S)=1 \qquad P(\phi)=0$

Mutually Exclusive

$$P(A\cup B)=P(A)+P(B) \quad \text{if} \quad AB=\phi$$

$$P(AB)=P(BA)$$

Conditional Probability and Independent Events

$$P(AB) = P(A)P(B|A)$$

$$\Rightarrow P(B|A) = \frac{P(AB)}{P(A)}$$

Similarly $$P(A|B) = \frac{P(AB)}{P(B)}$$

Independent Events

A and B are independent if

$$P(B|A) = P(B)$$

$$or \quad P(A|B) = P(A)$$

$$\Rightarrow P(AB) = P(A)P(B)$$

Does Order Matter in a Repeated Trials Series?

Let probability of success is p => probability of failure is q= 1-p

$$P(k \quad successes \quad in \quad a \quad specific \quad order \quad of \quad n \quad trials) = p^k (1-p)^{n-k}$$

$$P(k \quad successes \quad in \quad n \quad trials) = \binom{n}{k} p^k (1-p)^{n-k}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binary Symmetric Channel (BSC)

$$P(0|1) = P(1|0) = P_e$$

$$\text{and} \quad P(0|0) = P(1|1) = 1 - P_e$$

Two Interesting Examples from Digital Communication

- ***Concatenation increases probability of error***
  [start with probability of correct decision (1-P_E)]

- ***Redundancy reduces probability of error***
  [(start with probability of error P_E]

Random Variable

- ***Discrete Random Variables***
- ***Continuous Random Variables***

Let's first look at discrete random variables

1. ***Probabilities***
2. ***Joint Probabilities***
3. ***Conditional Probabilities***
4. ***Cumulative Distribution Function***

Discrete Probabilities

$$\sum_i P_x(x_i) = 1$$

Joint Probabilities

$$\sum_i \sum_j P_{xy}(x_i, y_j) = 1$$

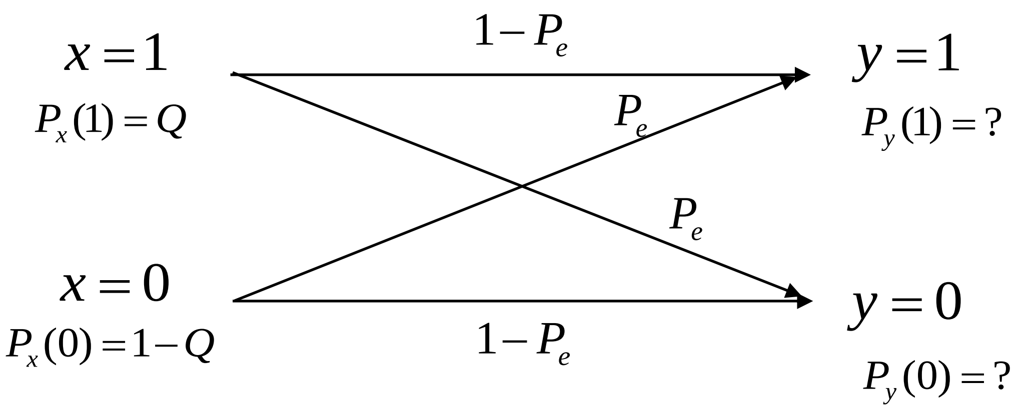
If x and y are independent

$$P_{xy}(x_i, y_j) = P_x(x_i) P_y(y_j)$$

Conditional Probabilities

$$\sum_i P_{x|y}(x_i | y_j) = \sum_j P_{y|x}(y_j | x_i) = 1$$

Binary Symmetric Channel (BSC)



We know

$$P_y(y_j)=\sum_i P_{xy}(x_i,y_j)$$

$$=\sum_i P_x(x_i)P_{y|x}(y_j|x_i)$$

$$\Rightarrow P_y(1)=P_x(0)P_{y|x}(1|0)+P_x(1)P_{y|x}(1|1)$$

$$=(1-Q)P_e+Q(1-P_e)$$

Cumulative Distribution Function (CDF)

$$F_X(x) = P(X \le x)$$

Properties

$$F_X(x) \ge 0$$

$$F_X(\infty) = 1$$

$$F_X(-\infty) = 0$$

$$F_X(x_1) \le F_X(x_2) \quad for \quad x_1 \le x_2$$

$$\frac{dF_X(x)}{dx} = P_X(x)$$

$$F_X(x) = \int_{-\infty}^{x} P_X(x)dx$$

We know that

$$F_X(\infty) = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} P_X(x)dx = 1$$

Similarly

$$P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$$

$$= \int_{-\infty}^{x_2} P_X(x)dx - \int_{-\infty}^{x_1} P_X(x)dx$$

$$= \int_{x_1}^{x_2} P_X(x)dx$$

Gaussian PDF

$$P_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

$$\Rightarrow F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-x^2}{2}} dx$$

Cannot be solved analytically, so let's define

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} e^{\frac{-x^2}{2}} dx$$ ***Which is widely tabulated***

$$\Rightarrow F_X(x) = 1 - Q(x)$$

More General Gaussian PDF

$$P_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-m)^2}{2\sigma^2}}$$

$$\Rightarrow F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-(x-m)^2}{2\sigma^2}} dx$$

Let $(x-m)/\sigma = z$

Then $$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m)/\sigma} e^{\frac{-z^2}{2}} dz$$

$$\Rightarrow F_X(x) = 1 - Q(\frac{x-m}{\sigma})$$

Statistical Averages (Means)

$$\overline{X} = \frac{N_1 x_1 + N_2 x_2 + \ldots N_n x_n}{N}$$

$$where \quad N = N_1 + N_2 + \ldots N_n = \sum_i N_i$$

$$\overline{X} = \frac{N_1 x_1}{N} + \frac{N_2 x_2}{N} + \ldots \frac{N_n x_n}{N} = \sum_{i=1}^{n} \frac{N_i}{N} x_i$$

$$= \sum_{i=1}^{n} P_x(x_i) x_i \quad as \quad N \to \infty$$

If X is continuous random variable

$$\overline{X} = \int_{-\infty}^{\infty} x P_X(x) dx$$

Mean of a Function of Random Variable

$$\text{Let} \quad Y = g(X)$$

$$\text{then} \quad \bar{Y} = \overline{g(X)} = \int_{-\infty}^{\infty} g(X) P_X(x) dx$$

Moments and Central Moments

nth moment

$$\overline{X^n} = \int_{-\infty}^{\infty} x^n P_X(x)dx$$

nth central moment

$$\overline{(X - \overline{X})^n} = \int_{-\infty}^{\infty} (x - \overline{x})^n P_X(x)dx$$

Special Cases

$$\text{variance} = \sigma_X^2 = \int_{-\infty}^{\infty} (x - \overline{x})^2 P_X(x)dx$$

$$\text{standard deviation} = \sigma_X = \sqrt{\int_{-\infty}^{\infty} (x - \overline{x})^2 P_X(x)dx}$$

Variance of Two Independent Variables

If X and Y are independent variables

$$Let \quad Z = X + Y$$

$$Then \quad \sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$$

$$\sigma_Z^2 = \overline{(z-\bar{z})^2} = \overline{[x+y-(\bar{x}+\bar{y})]^2}$$

$$= \overline{[(x-\bar{x})+(y-\bar{y})]^2}$$

$$= \overline{(x-\bar{x})^2} + \overline{(y-\bar{y})^2} + \overline{2(x-\bar{x})(y-\bar{y})}$$

$$\overline{2(x-\bar{x})(y-\bar{y})} = 0$$

$$= \sigma_X^2 + \sigma_Y^2$$

Central Limit Theorem

Sum of large number of random variables tend to be Gaussian.

$$Let \quad Z = X + Y$$

$$\Rightarrow \quad Y = Z - X$$

$$F_z(z) = P(z \le z) = P(x \le \infty, y \le z - x)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} P_{XY}(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} P_{XY}(x, y) dy$$

$$p_z(z) = \frac{dF_z(z)}{dz} \quad = \frac{d}{dz} \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} P_{XY}(x, y) dy$$

$$P_z(z) = \frac{dF_z(z)}{dz} \quad = \frac{d}{dz}\int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} P_{XY}(x,y)dy$$

$$= \int_{-\infty}^{\infty} dx \frac{d}{dz}\int_{-\infty}^{z-x} P_{XY}(x,y)dy$$

$$\Rightarrow P_Z(z) = \int_{-\infty}^{\infty} P_{XY}(x, z-x)dx$$

If X and Y are independent RVs, then

$$P_{XY}(x, z-x) = P_X(x)P_Y(z-x)$$

$$\Rightarrow P_Z(z) = \int_{-\infty}^{\infty} P_X(x)P_Y(z-x)dx$$

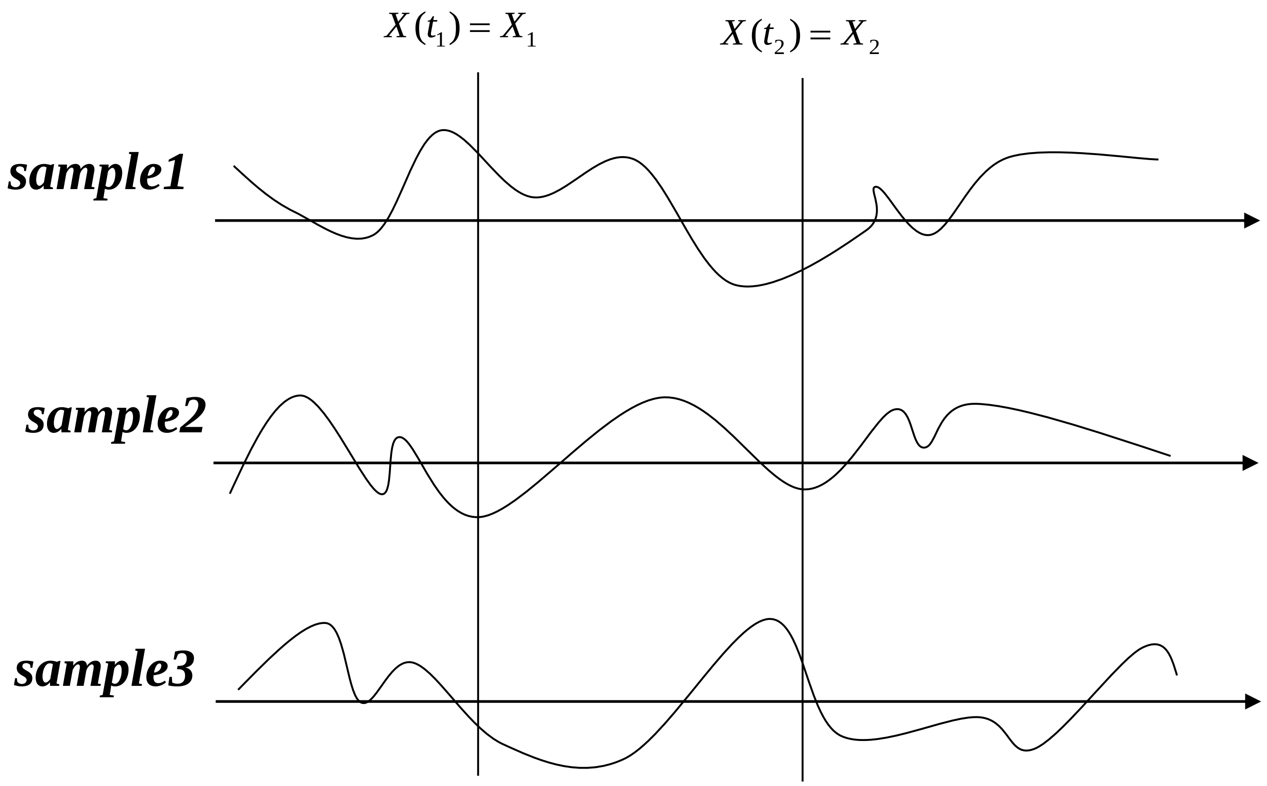
$$= P_X(x) * P_Y(y)$$

Random Processes

A random variable which is a function of time is a random process.

A random Variable is X and a random Process is X(t)

$$P_X(x;t)$$



Autocorrelation Function of A Random Process

$$R_X(t_1,t_2) = \overline{X(t_1)X(t_2)} = \overline{X_1 X_2}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 P_{X_1 X_2}(x_1, x_2)\, dx_1 dx_2$$

For stationary random processes:

$$P_X(x;t) = P_X(x)$$

and $$R_X(t_1,t_2) = R_X(t_2 - t_1) = R_X(\tau)$$

$$R_X(\tau) = \overline{X(t)X(t+\tau)}$$

An Example of a Random Process

$$X(t) = A\cos(\omega_c t + \Theta)$$

$$\overline{X(t)} = ? \quad and \quad R_X(\tau) = ?$$

$$\overline{X(t)} = \overline{A\cos(\omega_c t + \Theta)} = A\overline{\cos(\omega_c t + \Theta)}$$

$$\overline{\cos(\omega_c t + \Theta)} = \int_0^{2\pi} \cos(\omega_c t + \theta) P_\Theta(\theta) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(\omega_c t + \theta) d\theta = 0$$

$$\Rightarrow \overline{X(t)} = 0$$

$$R_X(t_1,t_2) = \overline{A^2\cos(\omega_c t_1+\Theta)\cos(\omega_c t_2+\Theta)}$$

$$= A^2\,\overline{\cos(\omega_c t_1+\Theta)\cos(\omega_c t_2+\Theta)}$$

$$= \frac{A^2}{2}\{\overline{\cos[\omega_c(t_1-t_2)]}+\overline{\cos[\omega_c(t_1+t_2)+2\Theta]}\}$$

$$= \frac{A^2}{2}\{\cos[\omega_c(t_1-t_2)]+\overline{\cos[\omega_c(t_1+t_2)+2\Theta]}\}$$

$$\overline{\cos[\omega_c(t_1+t_2)+2\Theta]} = \frac{1}{2\pi}\int_0^{2\pi}\cos[\omega_c(t_1+t_2)+2\theta)d\theta$$

$$= 0$$

$$\Rightarrow R_X(t_1,t_2) = \frac{A^2}{2}\cos\omega_c(t_1-t_2) = \frac{A^2}{2}\cos\omega_c(\tau)$$

Autocorrelation function of a power signal

$$\Re_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} g(t)g(t+\tau)dt$$

Reminder Slide

Same argument as in energy signals will prove that

$$\Re_g(\tau) = \Re_g(-\tau)$$

and $\Re_g(\tau) \Leftrightarrow S_g(\omega)$

also $S_y(\omega) = |H(\omega)|^2 S_g(\omega)$

if $Y(\omega) = H(\omega)G(\omega)$

Power Spectral Density of a Deterministic Signal

$$S_g(\omega) = \lim_{T\to\infty} \frac{|G_T(\omega)|^2}{T}$$

$$S_g(\omega) \Leftrightarrow R_g(\tau)$$

Power Spectral Density of Random Process

$$S_X(\omega) = \lim_{T\to\infty} \overline{\left[\frac{|X_T(\omega)|^2}{T} \right]}$$

$$S_X(\omega) \Leftrightarrow R_X(\tau)$$

$$S_X(\omega) \Leftrightarrow R_X(\tau)$$

$$\Rightarrow S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega(\tau)} d\tau$$

Also remember

$$R_X(\tau) = \overline{X(t)X(t+\tau)}$$

$$Let \quad t+\tau = \sigma$$

$$R_X(-\tau) = \overline{X(\sigma+\tau)X(\sigma)}$$

$$= \overline{X(\sigma)X(\sigma+\tau)} = R_X(\tau)$$

Power of a Random Process

$$P_X = \overline{X^2}$$

$$R_X(0) = \overline{X(t)X(t)}$$

$$= \overline{X^2(t)} = \overline{X^2}$$

$$\Rightarrow P_X = R_X(0)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} S_X(\omega)d\omega$$

Example 1 – Low-Pass Random Process

$$S_X(\omega) = \frac{N}{2} rect(\frac{\omega}{4\pi B})$$

$$\Rightarrow R_X(\tau) = NB \sin c(2\pi B\tau)$$

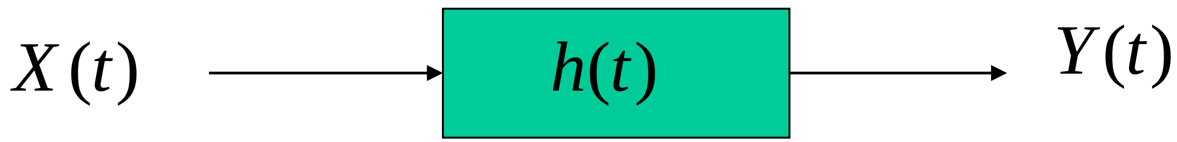
$$P_X = R_X(0) = NB$$

Alternatively

$$P_X = \frac{1}{\pi}\int_0^{\infty} S_X(\omega)d\omega = 2\int_0^{\infty} S_X(f)df$$

$$= 2\int_0^{B} \frac{N}{2} df = NB$$

Transmission through Linear System



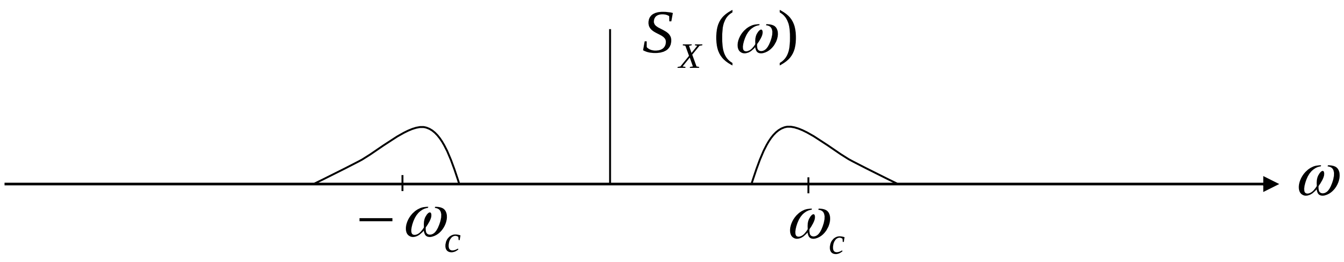
$$S_Y(\omega) = \left|H(\omega)\right|^2 S_X(\omega)$$

Sum of Independent Random Processes

$$If \qquad Z(t) = X(t) + Y(t)$$

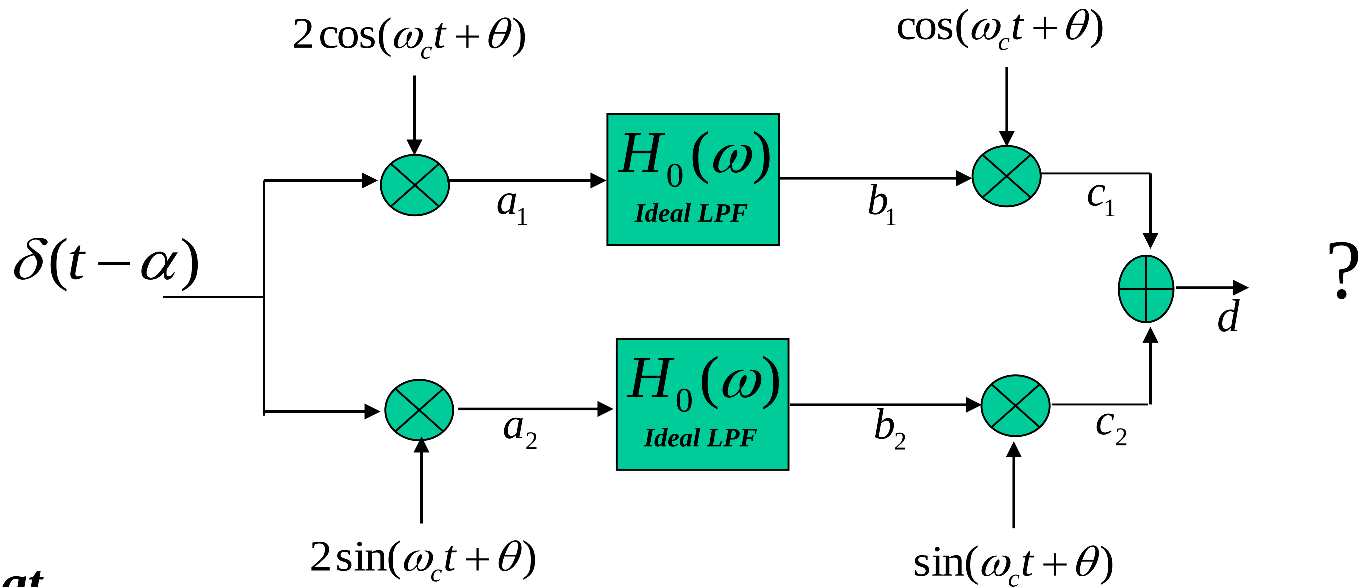
$$then \qquad S_Z(\omega) = S_X(\omega) + S_Y(\omega)$$

Bandpass Random Process



$$X(t) = X_c(t)\cos\omega_c t + X_s(t)\sin\omega_c t$$

Bandpass random process has two quadrature low pass components having equal power.



Signals at

$$a_1 = 2\cos(\omega_c\alpha+\theta)\delta(t-\alpha)$$

$$a_2 = 2\sin(\omega_c\alpha+\theta)\delta(t-\alpha)$$

$$b_1 = 2\cos(\omega_c\alpha+\theta)h_0(t-\alpha)$$

$$b_2 = 2\sin(\omega_c\alpha+\theta)h_0(t-\alpha)$$

$$c_1 = 2\cos(\omega_c\alpha+\theta)\cos(\omega_c t+\theta)h_0(t-\alpha)$$

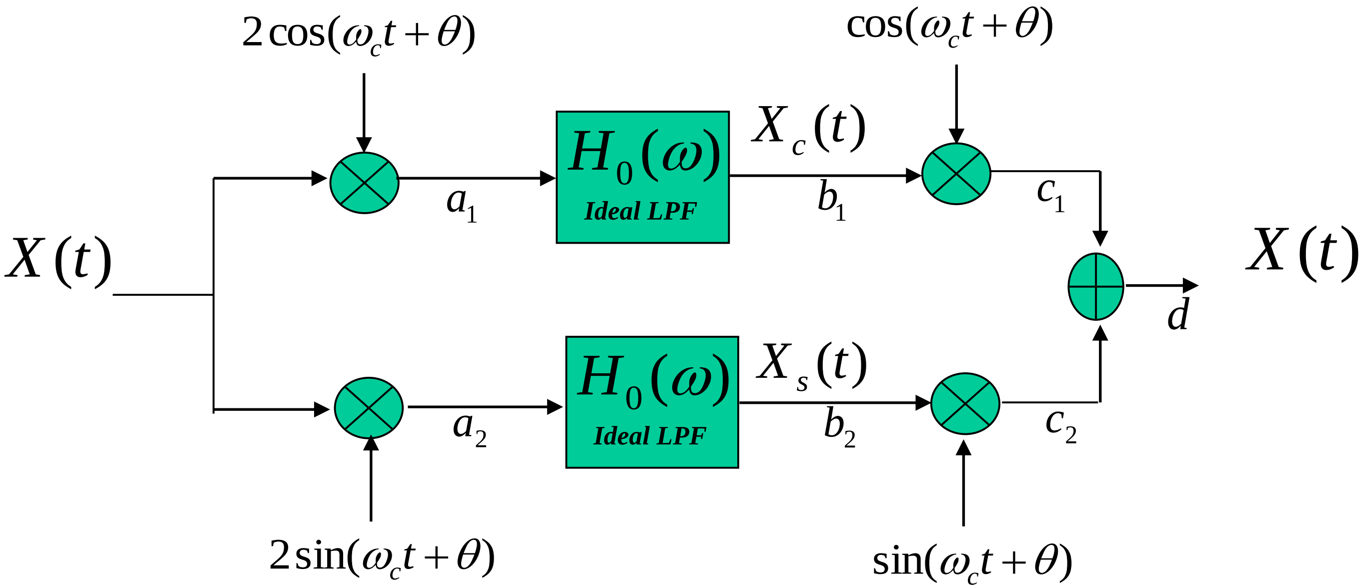
$$c_2 = 2\sin(\omega_c\alpha+\theta)\sin(\omega_c t+\theta)h_0(t-\alpha)$$

$$d = 2h_0(t-\alpha)[\cos(\omega_c\alpha+\theta)\cos(\omega_c t+\theta)+\sin(\omega_c\alpha+\theta)\sin(\omega_c t+\theta)]$$

$$= 2h_0(t-\alpha)\cos[\omega_c(t-\alpha)]$$

$$d = 2h_0(t-\alpha)\cos[\omega_c(t-\alpha)]$$

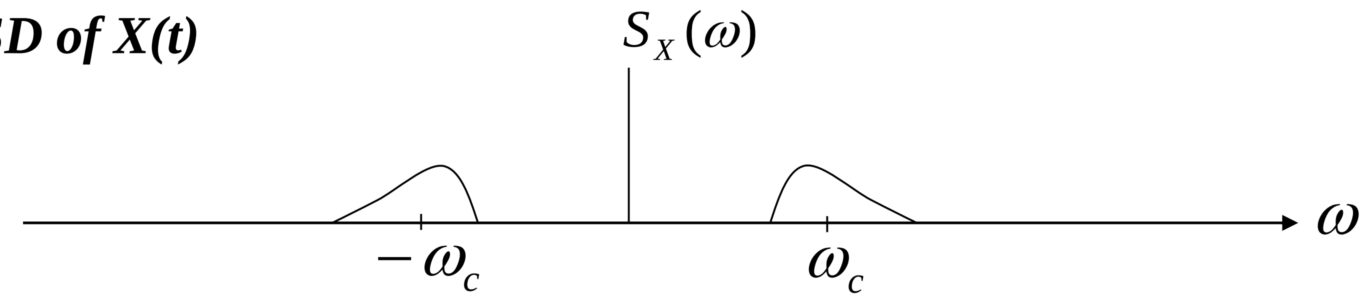
$$\Rightarrow H(\omega) = H_0(\omega+\omega_c) + H_0(\omega-\omega_c)$$



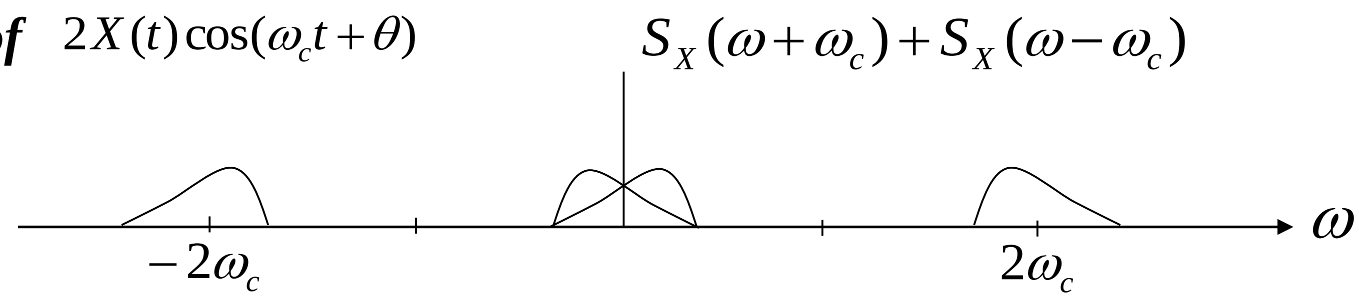
$$X(t) = X_c(t)\cos(\omega_c t+\theta) + X_s(t)\sin(\omega_c t+\theta)$$

PSD of Quadrature Components

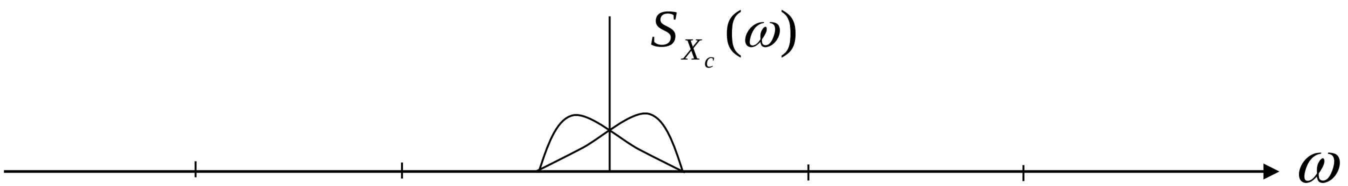
PSD of X(t)



PSD of $2X(t)\cos(\omega_c t+\theta)$



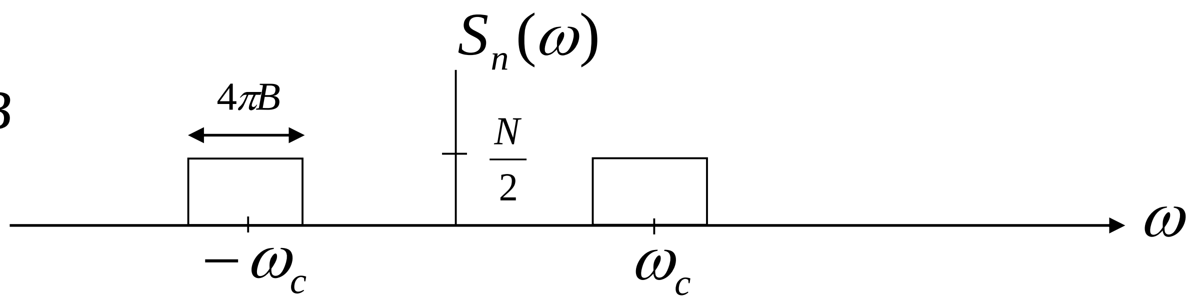
PSD of $2X(t)\cos(\omega_c t+\theta)$ ***after passing through LPF***



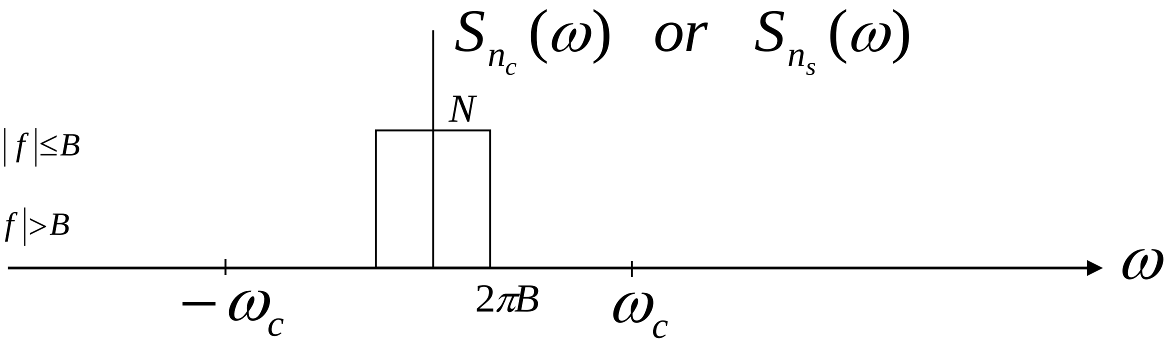
$$\Rightarrow \overline{X_c^2}=\overline{X_s^2}=\overline{X^2}$$

Example – Bandpass Noise

$$\overline{n^2} = 2\int_{f_c-B}^{f_c+B} \frac{N}{2} df = 2NB$$



$$S_{n_c}(f) = S_{n_s}(f) = \begin{cases} N & |f| \le B \\ 0 & |f| > B \end{cases}$$



$$\overline{n_c^2} = \overline{n_s^2} = 2\int_0^B N df = 2NB$$

$$\Rightarrow \overline{n^2} = \overline{n_c^2} = \overline{n_s^2} = 2NB$$