From Fourier Series towards Fourier Transform

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(t) e^{-jn\omega_0 t} dt$$

$$D_n = \frac{1}{T_0} \int_{-\infty}^{\infty} g(t) e^{-jn\omega_0 t} dt, \quad when \quad \lim_{T_0 \to \infty}$$

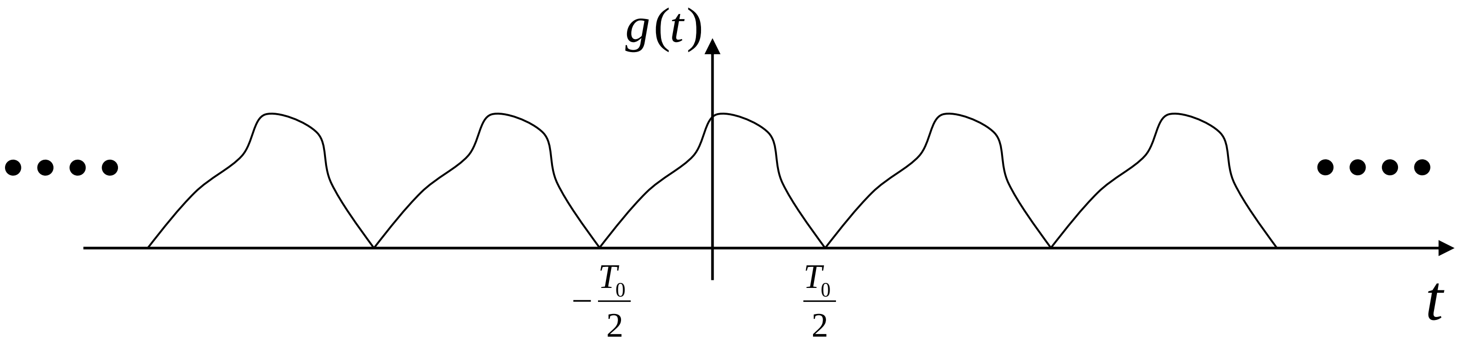
$$D_n = \frac{1}{T_0} \int_{-\infty}^{\infty} g(t) e^{-jn\omega_0 t} dt, \quad when \quad \lim_{T_0 \to \infty}$$

Let's Consider a function

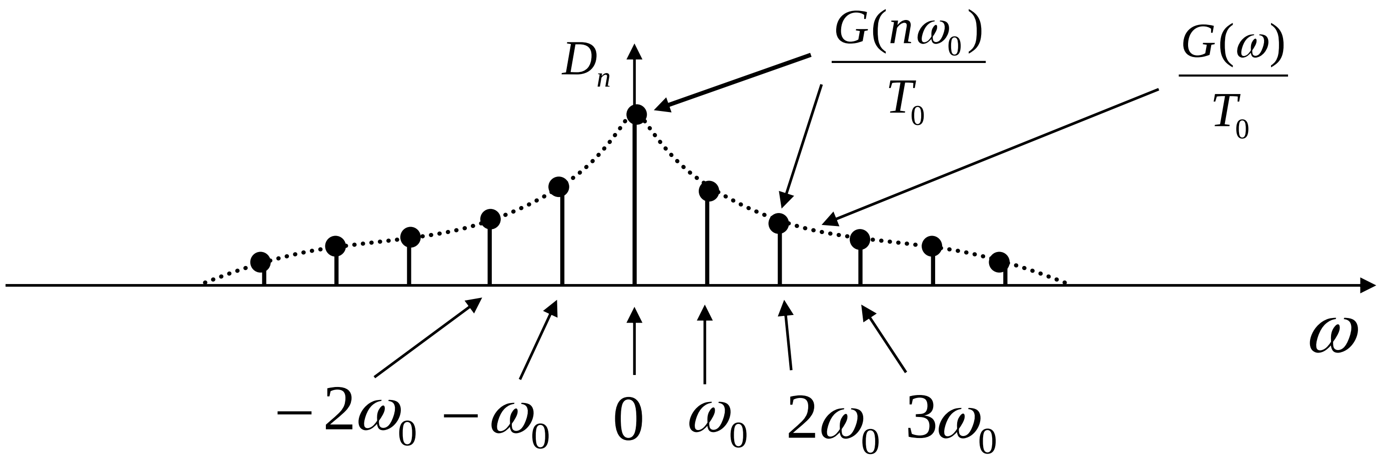
$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

We can express D_n in terms of G(w)

$$D_n = \frac{1}{T_0} G(n\omega_0)$$

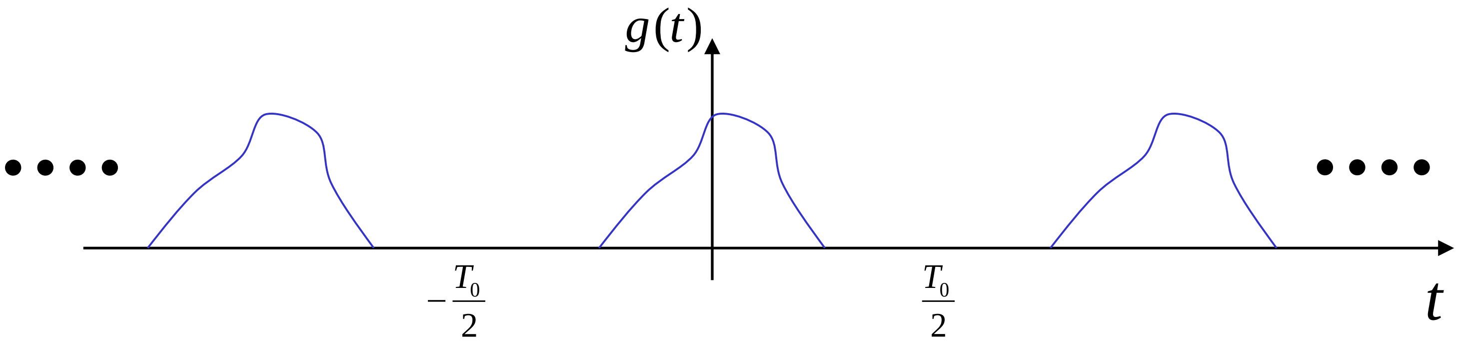


$g(t)$
....
....
$-\frac{T_0}{2}$
$\frac{T_0}{2}$
t

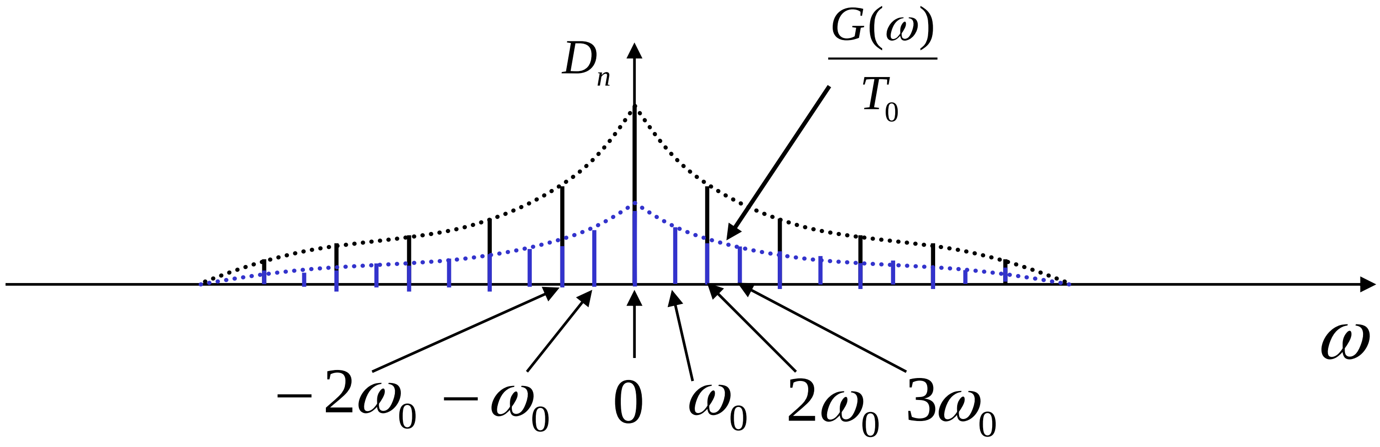


D_n
$\frac{G(n\omega_0)}{T_0}$
$\frac{G(\omega)}{T_0}$
ω
$-2\omega_0$
$-\omega_0$
0
ω_0
$2\omega_0$
$3\omega_0$

$2xT_0$

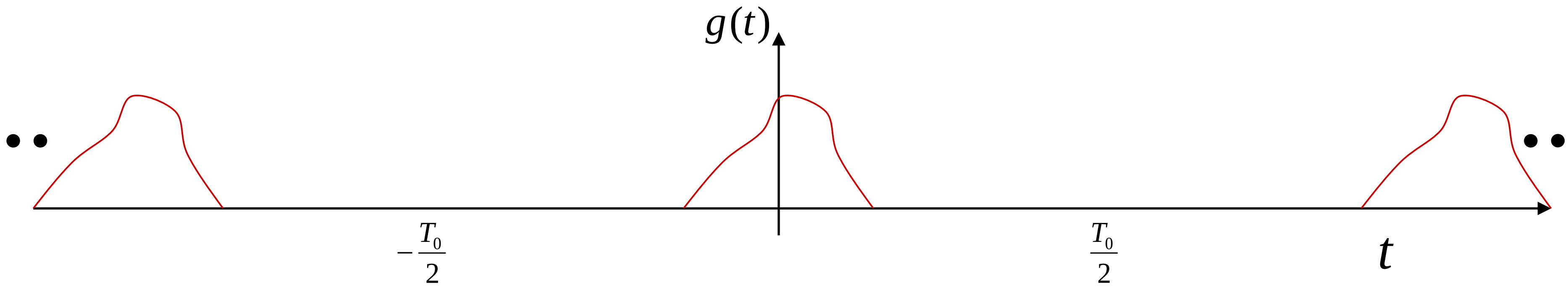


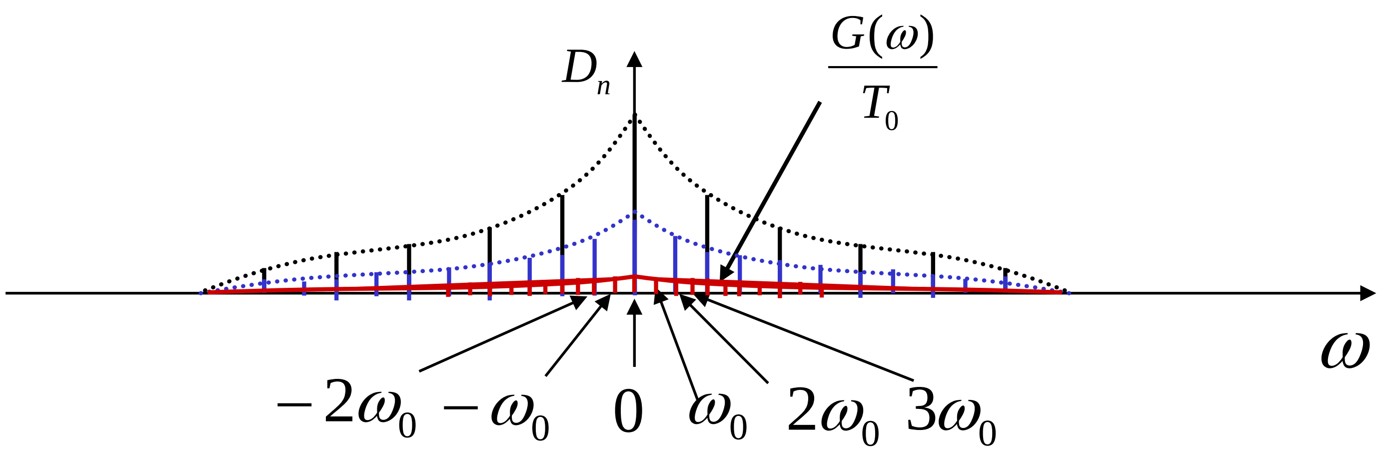
g(t)
....
....
$-\frac{T_0}{2}$
$\frac{T_0}{2}$
t



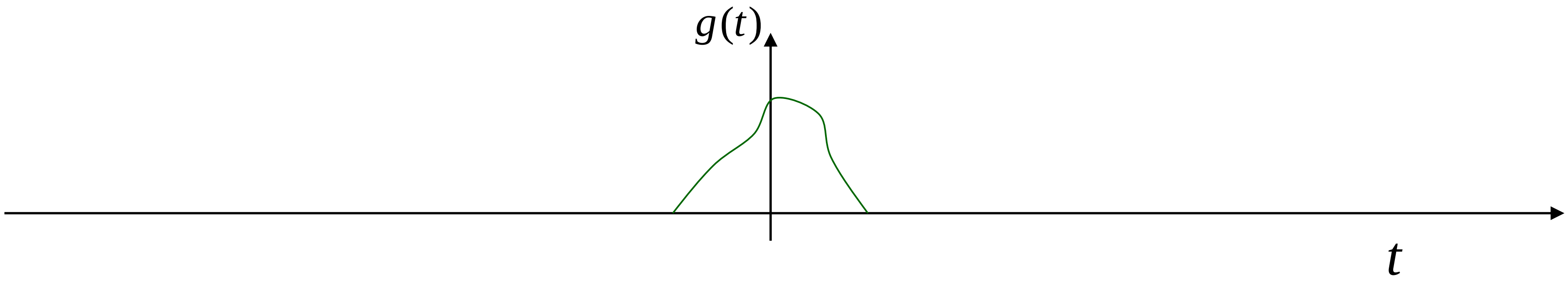
D_n
$\frac{G(\omega)}{T_0}$
ω
$-2\omega_0$
$-\omega_0$
0
ω_0
$2\omega_0$
$3\omega_0$

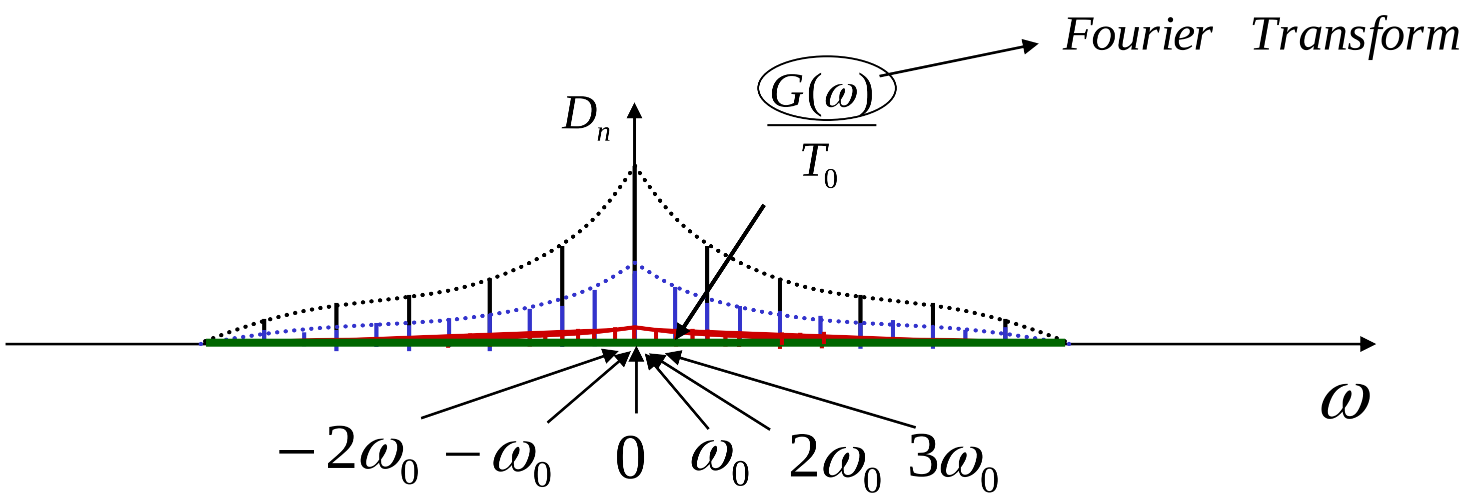
$2x2xT_0$


$g(t)$
$-\frac{T_0}{2}$
$\frac{T_0}{2}$
t


D_n
$\frac{G(\omega)}{T_0}$
ω
$-2\omega_0$
$-\omega_0$
0
ω_0
$2\omega_0$
$3\omega_0$

$\infty x T_0$





$$\omega_0 = \Delta\omega$$

Fourier Transform of a Singal g(t) $G(\omega) = F[g(t)]$

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

Please note that this does not apply only to aperiodic signals, it can apply to any signal, periodic or aperiodic

Now Let's talk about the Inverse Fourier Transform

$$g(t) = F^{-1}[G(\omega)]$$

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{G(n\omega_0)}{T_0} e^{jn\omega_0 t}$$

$$\lim_{T_0 \to \infty} \quad \omega_0 \to \Delta\omega$$

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)}{T_0} e^{jn\Delta\omega t}$$

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)\Delta\omega}{2\pi} e^{jn\Delta\omega t}$$

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\Delta\omega}$$

$$g(t)=\sum_{n=-\infty}^{\infty}\frac{G(n\Delta\omega)\Delta\omega}{2\pi}e^{jn\Delta\omega t}$$

$$T_0=\frac{2\pi}{\omega_0}=\frac{2\pi}{\Delta\omega}$$

$$g(t)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}G(n\Delta\omega)\Delta\omega e^{jn\Delta\omega t}$$

$$g(t)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}G(n\Delta\omega)e^{jn\Delta\omega t}\Delta\omega$$

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\omega)e^{j\omega t}d\omega$$

$$g(t)=F^{-1}[G(\omega)]$$

Inverse Fourier Transform

Fourier Transform of a Singal g(t)

$$G(\omega) = F[g(t)]$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

Inverse Fourier Transform

$$g(t) = F^{-1}[G(\omega)]$$

$$g(t) = \frac{1}{2\pi} \int_{\infty}^{\infty} G(\omega) e^{j\omega t} d\omega$$

Summary Charts Ahead

$$g(t) \approx cx(t)$$

$$c = \frac{1}{E_x}\int_{t_1}^{t_2} g(t)x(t)dt$$

g(t) and x(t) are orthogonal if

$$\int_{t_1}^{t_2} g(t)x(t)dt = 0$$

or if x(t) is complex

$$\int_{t_1}^{t_2} g(t)x^*(t)dt = 0$$

If a set of functions x(t) exists

$$\int_{t_1}^{t_2} x_m(t)x_n^*(t)dt = \begin{cases} 0 & \text{if } m \neq n \\ E_n & \text{if } m = n \end{cases}$$

then

$$g(t) = \sum_{n=-\infty}^{\infty} c_n x_n(t) \qquad t_1 \le t \le t_2$$

$$c_n = \frac{1}{E_n}\int_{t_1}^{t_2} g(t)x_n^*(t)dt$$

If a set of functions x(t) exists

$$\int_{t_1}^{t_2} x_m(t)x_n^*(t)dt = \begin{cases} 0 & \text{if } m \neq n \\ E_n & \text{if } m = n \end{cases}$$

then

$$g(t) = \sum_{n=-\infty}^{\infty} c_n x_n(t) \qquad t_1 \le t \le t_2$$

$$c_n = \frac{1}{E_n}\int_{t_1}^{t_2} g(t)x_n^*(t)dt$$

$$1,\ \cos(n\omega_0 t) \qquad 1 \le n \le \infty$$

$$\sin(n\omega_0 t)$$

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)$$

$$+ \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$-\frac{T_0}{2} \le t \le \frac{T_0}{2}\ {}_2$$

$$a_0 = \frac{1}{T_0}\int_{T_0} g(t)dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} g(t)\cos(n\omega_0 t)dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} g(t)\sin(n\omega_0 t)dt$$

If a set of functions x(t) exists

$$\int_{t_1}^{t_2} x_m(t)x_n^*(t)dt = \begin{cases} 0 & if\ m \neq n \\ E_n & if\ m = n \end{cases}$$

then

$$g(t) = \sum_{n=-\infty}^{\infty} c_n x_n(t) \qquad t_1 \leq t \leq t_2$$

$$c_n = \frac{1}{E_n}\int_{t_1}^{t_2} g(t)x_n^*(t)dt$$

$$1,\ \cos(n\omega_0 t) \qquad 1 \leq n \leq \infty$$

$$\sin(n\omega_0 t)$$

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$C_0 = a_0$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

If a set of functions x(t) exists

$$\int_{t_1}^{t_2} x_m(t)x_n^*(t)dt = 0 \quad \begin{cases} \text{if } m \neq n \\ \text{if } m = n \end{cases}$$

then

$$g(t) = \sum_{n=-\infty}^{\infty} c_n x_n(t) \qquad t_1 \le t \le t_2$$

$$c_n = \frac{1}{E_n}\int_{t_1}^{t_2} g(t)x_n^*(t)dt$$

$$e^{jn\omega_0 t} \qquad -\infty \le n \le \infty$$

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{1}{T_0}\int_{T_0} g(t)e^{-jn\omega_0 t}dt$$

$$|D_n| = \left|\frac{C_n}{2}\right|$$

$$\angle D_n = \theta_n$$

Fourier Transform of a Singal g(t)

$$D_n = \frac{1}{T_0}\int_{T_0} g(t)e^{-jn\omega_0 t}dt \Rightarrow$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt$$

Inverse Fourier Transform

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \Rightarrow$$

$$g(t) = \frac{1}{2\pi}\int_{\infty}^{\infty} G(\omega)e^{j\omega t}d\omega$$

$$g(t) \Leftrightarrow G(\omega)$$

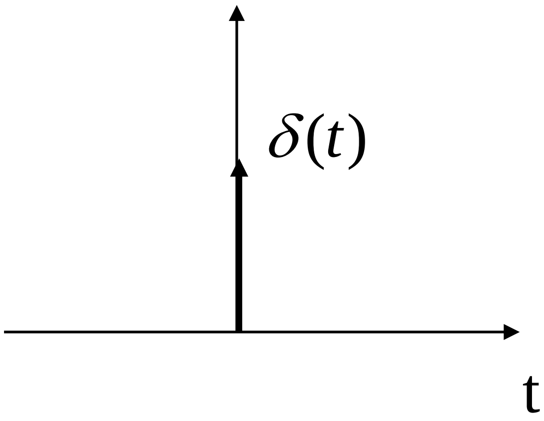
Some Examples of Fourier Transform

1. ***An impulse function***
2. ***A constant function (via inverse transform)***
3. ***Complex exponential function (via inverse transform)***
4. ***Sinosoidal Function***
5. ***Rectangular Pulse Signal***

Fourier Transform of Impulse Function

Definition:

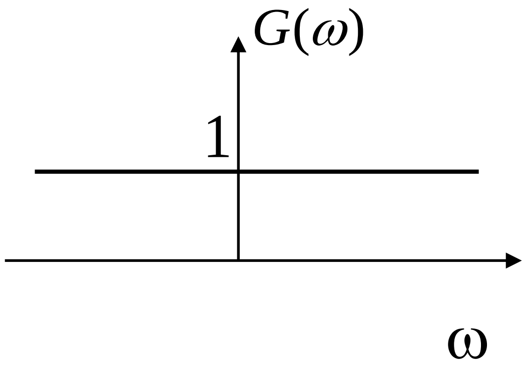
$$\begin{cases} \delta(t) = 0 \quad t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t) = 1 \end{cases}$$



Fourier Transform of a Signal g(t)

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$\Rightarrow G(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{0} = 1$$

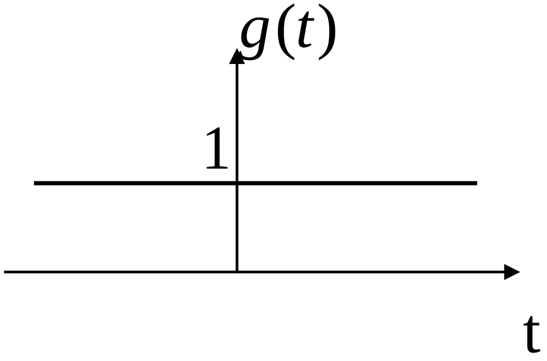


Fourier Transform of a Constant Function

Fourier Transform of a Signal g(t)

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dt \quad = \left| \frac{e^{-j\omega t}}{-j\omega} \right|_{-\infty}^{\infty} = 0 + \infty = \infty$$

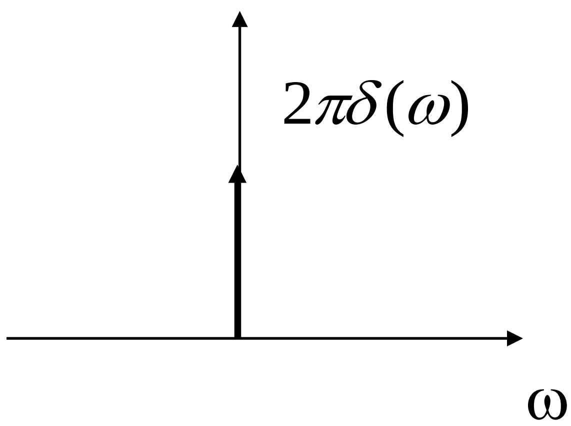


Let's try indirectly – let's find the Inverse Fourier Transform of δ(ω)

$$g(t) = F^{-1}[G(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega$$

$$\Rightarrow F^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{0} = \frac{1}{2\pi}$$

$$\Rightarrow F[\frac{1}{2\pi}] = \delta(\omega) \qquad \Rightarrow F[1] = 2\pi\delta(\omega)$$



Fourier Transform of a Complex Exponential

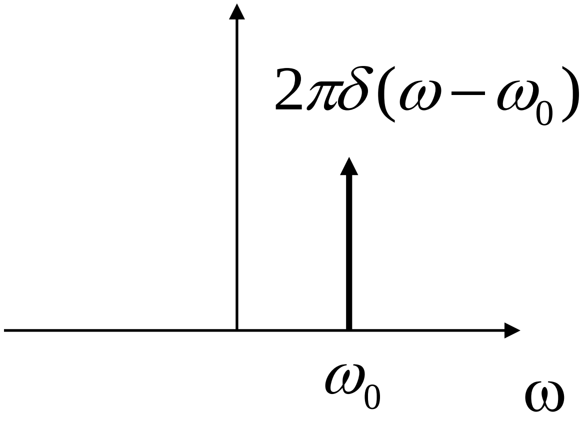
Let's find the Inverse Fourier Transform of $\delta(\omega-\omega_0)$

$$g(t) = F^{-1}[G(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega$$

$$\Rightarrow F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

$$\Rightarrow F[\frac{e^{j\omega_0 t}}{2\pi}] = \delta(\omega-\omega_0)$$

$$\Rightarrow F[e^{j\omega_0 t}] = 2\pi\delta(\omega-\omega_0)$$

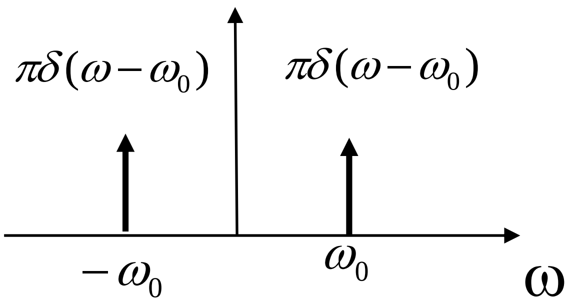


Fourier Transform of a Sinusoidal Signal

$$g(t) = \cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\Rightarrow F[\cos(\omega_0 t)] = F[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}] = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$= \pi[\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)]$$



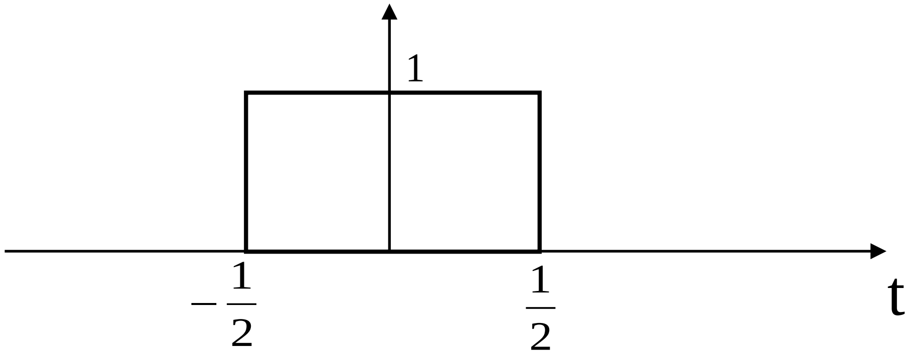
$$x(t) = \sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\Rightarrow F[\sin(\omega_0 t)] = F[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}] = -j\pi\delta(\omega - \omega_0) + \pi j\delta(\omega + \omega_0)$$

$$= j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

Fourier Transform of a Rectangular Function

$$rect(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| > 1/2 \end{cases}$$



$$F[g(t)] = G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$\Rightarrow F[rect(t)] = \int_{-1/2}^{1/2} e^{-j\omega t} dt \quad = \left| \frac{e^{-j\omega t}}{-j\omega} \right|_{-1/2}^{1/2} = \frac{e^{-j\omega/2}}{-j\omega} - \frac{e^{j\omega/2}}{-j\omega}$$

$$= \frac{e^{j\omega/2}}{j\omega} - \frac{e^{-j\omega/2}}{j\omega} = \frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega}$$

$$F[rect(t)] = \frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega}$$

$$= \frac{2}{\omega} \frac{[e^{j\omega/2} - e^{-j\omega/2}]}{2j}$$

$$= \frac{2}{\omega} \sin(\omega/2) = \frac{\sin(\omega/2)}{(\omega/2)}$$

Definition

$$\sin c(t) = \frac{\sin(t)}{t}$$

or

$$\sin c(\omega) = \frac{\sin(\omega)}{\omega}$$

$$= \sin c(\omega/2)$$

Some Properties of Fourier Transform

Linearity

$$k_1 g_1(t) + k_2 g_2(t) \Leftrightarrow k_1 G_1(\omega) + k_2 G_2(\omega)$$

Symmetry

$$g(t) \Leftrightarrow G(\omega)$$

$$\Downarrow$$

$$G(t) \Leftrightarrow 2\pi g(-\omega)$$

Scaling

$$g(t) \Leftrightarrow G(\omega)$$

$$\Downarrow$$

$$g(at) \Leftrightarrow \frac{1}{|a|} G(\frac{\omega}{a})$$

Some Properties of Fourier Transform --- cont.

Time Shifting – linear phase shift

$$g(t-t_0) \Leftrightarrow G(\omega)e^{-j\omega t_0}$$

Frequency Shifting - modulation

$$g(t)e^{j\omega_0 t} \Leftrightarrow G(\omega-\omega_0)$$

Some Properties of Fourier Transform --- cont.

Convolution - definition $$g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau) w(t-\tau) d\tau$$

If $g_1(t) \Leftrightarrow G_1(\omega)$ ***and*** $w(t) \Leftrightarrow W(\omega)$

$$\Downarrow$$

Time Convolution $$g(t) * w(t) \Leftrightarrow G(\omega) W(\omega)$$

Frequency Convolution $$g(t) w(t) \Leftrightarrow \frac{1}{2\pi} G(\omega) * W(\omega)$$

Some Properties of Fourier Transform --- cont.

Time Differentiation and Integration

$$g(t) \Leftrightarrow G(\omega)$$

$$\Downarrow$$

Differentiation

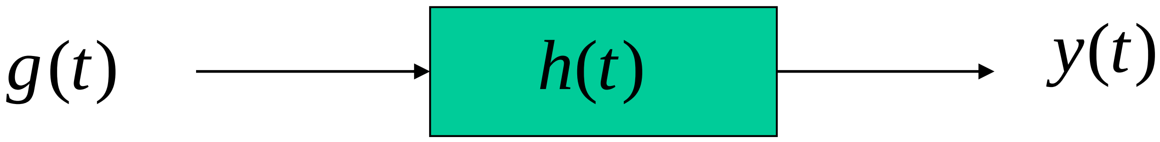
$$\frac{dg}{dt} \Leftrightarrow j\omega G(\omega)$$

Integration

$$\int_{-\infty}^{t} g(\tau)d\tau \Leftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

Note: Equivalent to convolving with a unit step function

Signal Transmission through a Linear Channel



$h(t)$ - ***Impulse Response of a Linear Time Invariant System***

$$y(t) = g(t) * h(t)$$

$$\Rightarrow \quad Y(\omega) = G(\omega)H(\omega) \qquad\qquad h(t) \Leftrightarrow H(\omega)$$

Ideal Distortionless Transmission

$$y(t) = kg(t - t_d)$$

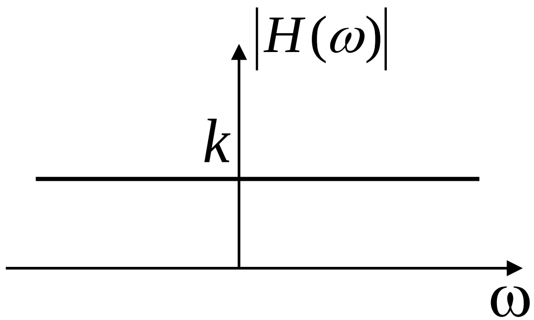
$$\Rightarrow \quad Y(\omega) = kG(\omega)e^{-j\omega t_d}$$

we know $$Y(\omega) = G(\omega)H(\omega)$$

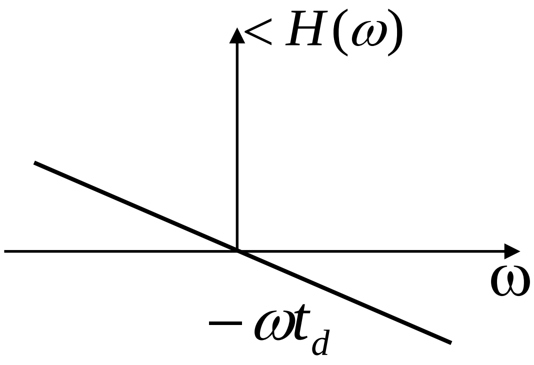
$$\Rightarrow \quad H(\omega) = ke^{-j\omega t_d}$$

$$\Rightarrow \qquad H(\omega) = ke^{-j\omega t_d}$$

Amplitude Response



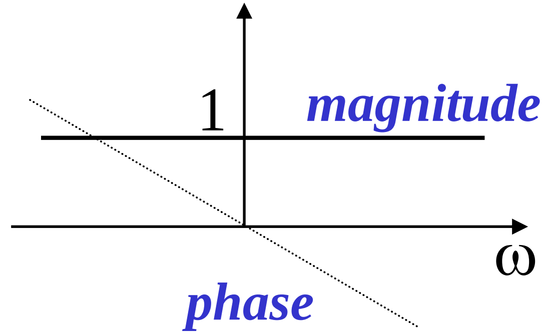
Phase Response



Ideal vs. Non Ideal Filters

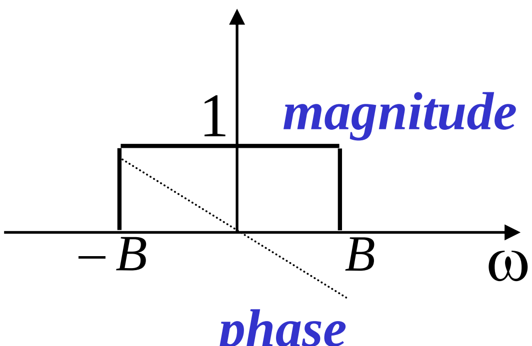
Ideal Filter

$$H(\omega) = e^{-j\omega t_d}$$



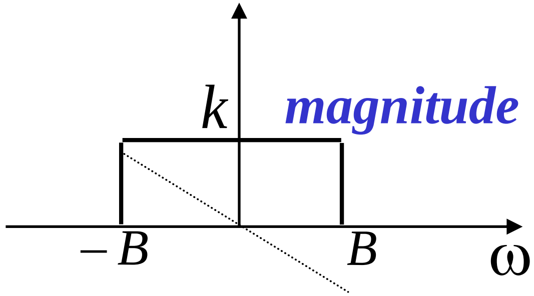
Non Ideal Filter

$$H(\omega) = rect(\frac{\omega}{2B})e^{-j\omega t_d}$$



Non Ideal Filter

$$H(\omega) = rect(\frac{\omega}{2B})e^{-j\omega t_d}$$



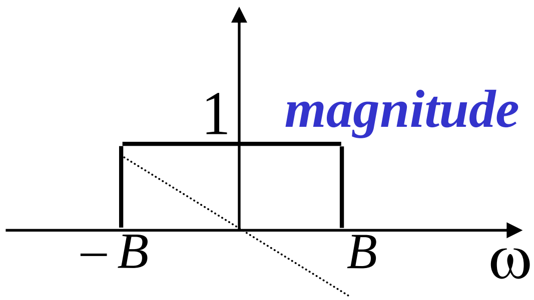
Impulse Response $h(t) = ?$

$$h(t) = F^{-1}\left[rect(\frac{\omega}{2B})e^{-j\omega t_d} \right]$$

$$= \frac{B}{\pi} \sin c[B(t - t_d)]$$

Non Ideal Filter

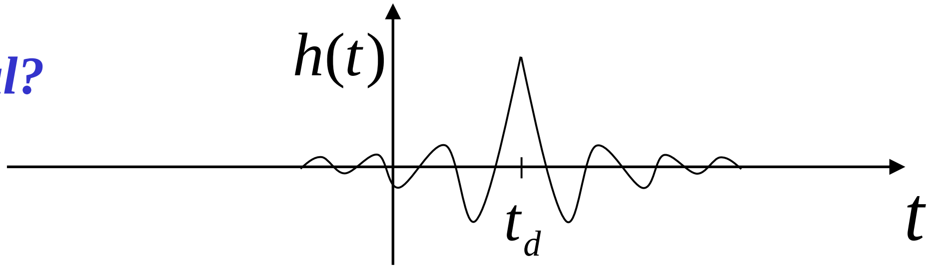
$$H(\omega) = rect(\frac{\omega}{2B})e^{-j\omega t_d}$$



Impulse Response

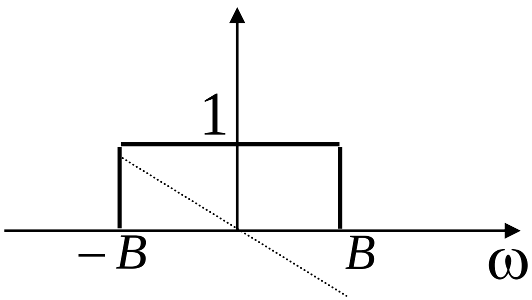
$$h(t) = \frac{B}{\pi}\sin c[B(t-t_d)]$$

Is this practical?

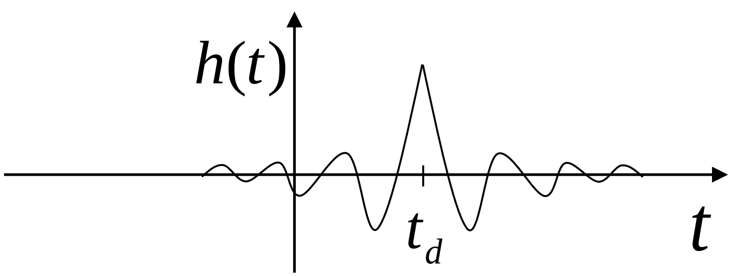


Non Ideal Filter

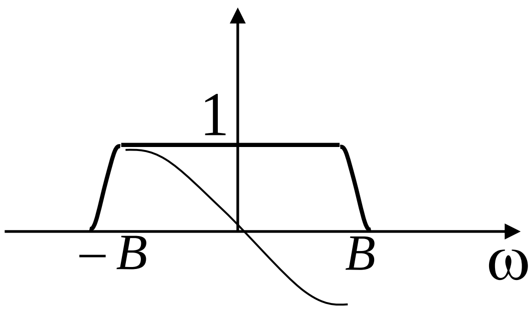
$$H(\omega) = rect(\frac{\omega}{2B})e^{-j\omega t_d}$$

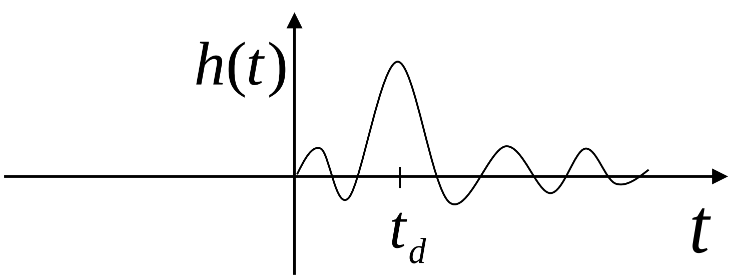


$$h(t) = \frac{B}{\pi}\sin c[B(t - t_d)]$$



Practical Filter

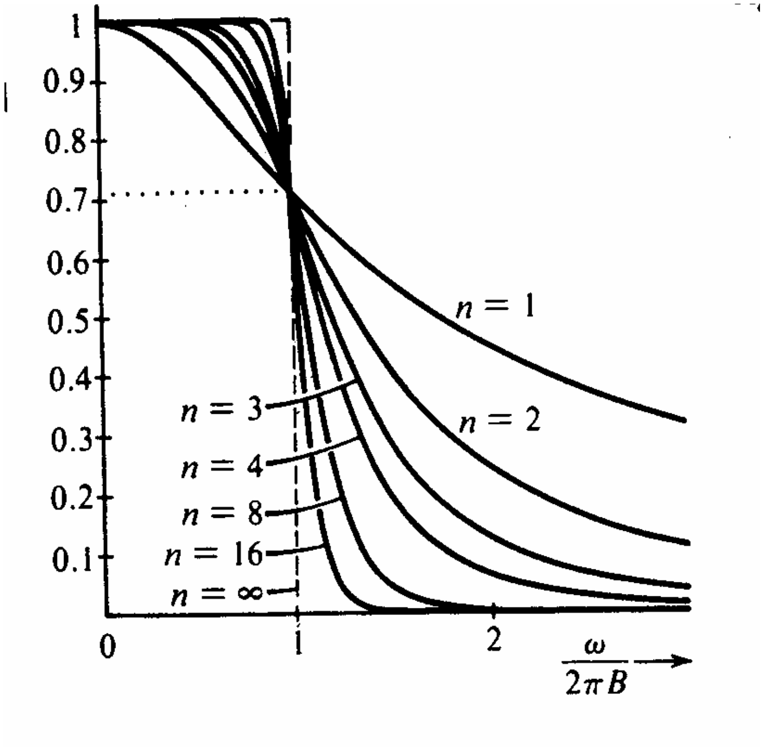




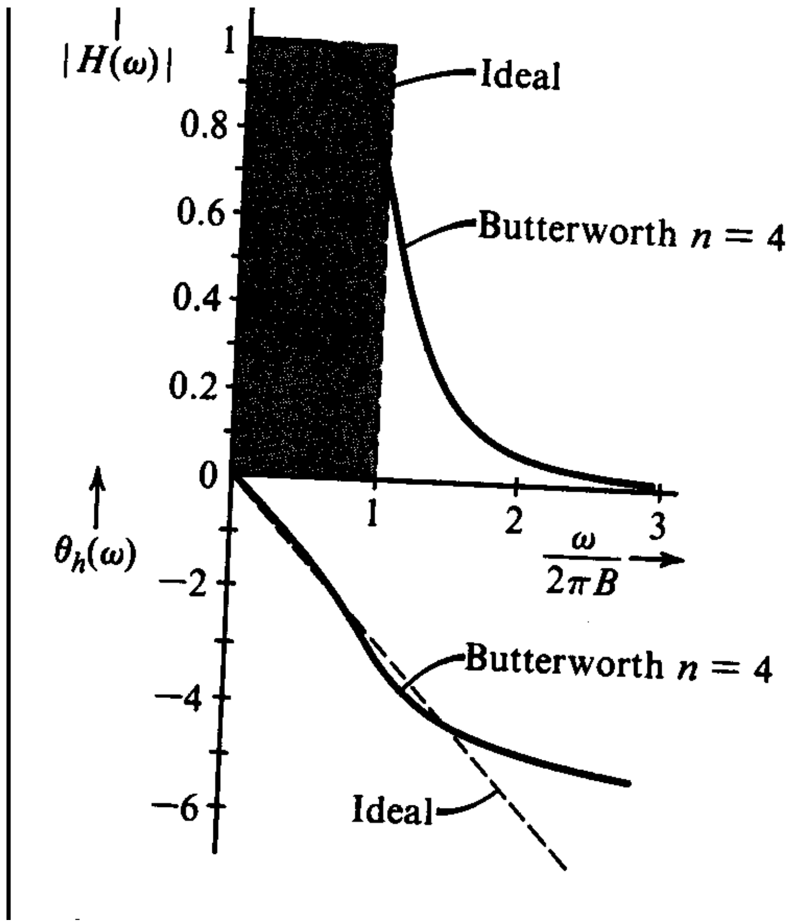
An Example of Practical Filter

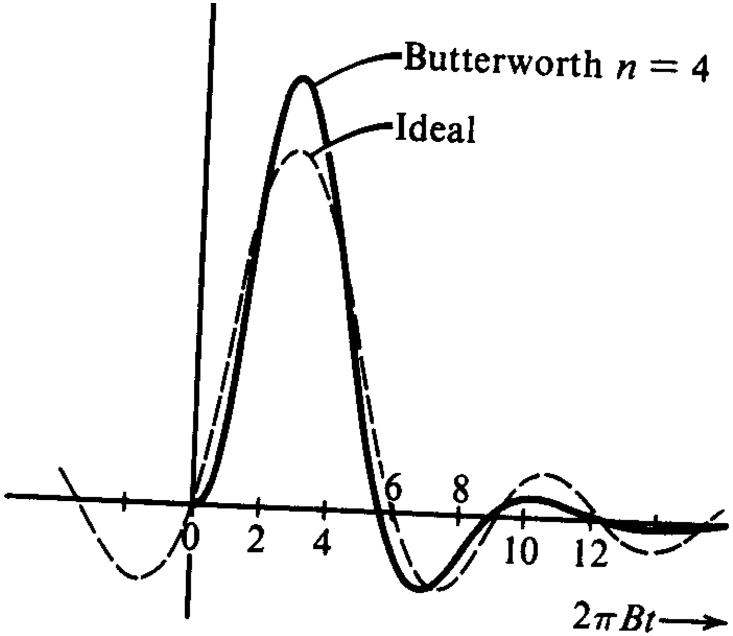
$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega/2\pi B)^{2n}}}$$

$|H(\omega)|$



An Example with n=4





Energy of a Signal and Parseval's Theorem

$$E_g = \int_{-\infty}^{\infty} g(t)g^*(t)dt$$

$$E_g = \int_{-\infty}^{\infty} g(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\omega)e^{-j\omega t}d\omega\right]dt$$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\omega)\left[\int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt\right]d\omega$$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\omega)G(\omega)d\omega$$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|G(\omega)\right|^2 d\omega$$

Energy Spectral Density

$$E_g = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| G(\omega) \right|^2 d\omega$$

$$= \int_{-\infty}^{\infty} \left| G(2\pi f) \right|^2 df$$

$$\Downarrow$$

$$\Psi_g = \left| G(2\pi f) \right|^2$$ ***energy per unit bandwidth (Hz)***

$$= \left| G(\omega) \right|^2$$