The Hot air balloon problem

Here’s a problem that brings together concepts regarding buoyancy, force and acceleration, and the ideal gas law. I suggest you have a hot air balloon, which has a mass of 700kg uninflated, the balloon, when inflated, has a volume of 2900m³. The air inside the balloon has $T_{in}=120^\circ C$, and outside, it’s $T_{out}=20^\circ C$. The pressure inside and outside the balloon is atmospheric pressure ($1.01 \times 10^5 \text{ N/m}^2$). Assume the molar mass of the air is 28g/mol. I wanted to know:

a. The initial rate of acceleration;
b. Outside temperature at which the balloon would be neutrally buoyant;
c. How much heat would be needed to heat up the air (at constant pressure).

For (a). Clearly, to find acceleration, we will need to find the forces acting on the balloon. The only forces acting on it will be gravity (of course) and a buoyancy force. Either way, we’re going to need to determine the mass of the air inside the balloon- to do that, we need the density. We have enough info to do just that, with a slight adjustment of the ideal gas law. I can rewrite the ideal gas law

$$pV = nRT$$

in terms of mass and molar mass:

$$pV = \frac{m}{M}RT$$

And, with a little re-arranging:

$$pM = \rho RT$$

Hence:

$$\rho = \frac{pM}{RT} = \left(\frac{1.01 \times 10^5 \text{ N/m}^2}{(8.31 \text{ J/mol/K})(393K)}\right)(0.028 \text{ kg/mol}) = 0.868 \text{ kg/m}^3$$

Therefore, the total mass of the inflated balloon is

$$m_{tot} = 700 \text{ kg} + \rho_{in} V = 700 \text{ kg} + (0.868 \text{ kg/m}^3)(2900 \text{ m}^3) = 3218 \text{ kg}$$

Note that the air inside the balloon weighs a lot more than the balloon itself!

At this stage, I can write the force balance in the vertical direction:

$$\Sigma F = B - m_{tot}g = \rho_{outside} V g - m_{tot}g = m_{tot}a$$

Or, solving for the acceleration:

$$a = \frac{\rho_{outside} V g - m_{tot}g}{m_{tot}} = 0.453 \text{ m/s}^2.$$

For the balloon to be neutrally buoyant, i.e. the buoyancy force and the gravity force are equal,

$$\Sigma F = B - m_{tot}g = \rho_{outside} V g - m_{tot}g = m_{tot}a = 0$$

Or

$$\rho_{outside} V g = m_{tot}g$$

$$\rho_{outside} V = m_{tot}$$

$$\rho_{outside} = \frac{m_{tot}}{V} = \frac{3218 \text{ kg}}{2900 \text{ m}^3} = 1.11 \text{ kg/m}^3$$
I can use this density in the ideal gas law to get the temperature where the balloon is neutrally buoyant
\[ pM = \rho_{NB}RT_{NB} \]
\[ T_{NB} = \frac{pM}{\rho_{NB}R} = 306.5K \]

Finally, how much energy does it take to heat up the air? We assume the change in temperature occurs at constant pressure:
\[ Q = nC_p\Delta T = n(C_v + R)\Delta T \]
So we need the number of moles of substance:
\[ n = \frac{m}{M} = \frac{(2517\text{kg})}{0.028\text{kg/mol}} = 89900\text{mol} \]
Therefore
\[ Q = n(C_v + R)\Delta T = (89900\text{mol})(20.8 + 8.31\text{J/mol/K})(100\text{K}) = 262\text{MJ} \]