19 Vector Spaces

Still round the corner there may wait
A new road or a secret gate.

J. R. R. Tolkien, *The Fellowship of the Ring*

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

David Hilbert (1862–1943)

Definition and Examples

Abstract algebra has three basic components: groups, rings, and fields. Thus far we have covered groups and rings in some detail, and we have touched on the notion of a field. To explore fields more deeply, we need some rudiments of vector space theory that are covered in a linear algebra course. In this chapter, we provide a concise review of this material.

Definition Vector Space

A set V is said to be a *vector space* over a field F if V is an Abelian group under addition (denoted by $+$) and, if for each $a \in F$ and $v \in V$, there is an element av in V such that the following conditions hold for all a, b in F and all u, v in V.

1. $a(v + u) = av + au$
2. $(a + b)v = av + bv$
3. $a(bv) = (ab)v$
4. $1v = v$

The members of a vector space are called *vectors*. The members of the field are called *scalars*. The operation that combines a scalar a and a vector v to form the vector av is called *scalar multiplication*. In general, we will denote vectors by letters from the end of the alphabet, such as u, v, w, and scalars by letters from the beginning of the alphabet, such as a, b, c.

EXAMPLE 1 The set $\mathbf{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbf{R}\}$ is a vector space over $\mathbf{R}$. Here the operations are the obvious ones:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and

$$b(a_1, a_2, \ldots, a_n) = (ba_1, ba_2, \ldots, ba_n).$$

EXAMPLE 2 The set $M_2(Q)$ of 2×2 matrices with entries from Q is a vector space over Q. The operations are

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$$

and

$$b\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} ba_1 & ba_2 \\ ba_3 & ba_4 \end{bmatrix}.$$

EXAMPLE 3 The set $Z_p[x]$ of polynomials with coefficients from Z_p is a vector space over Z_p, where p is a prime.

EXAMPLE 4 The set of complex numbers $\mathbf{C} = \{a + bi \mid a, b \in \mathbf{R}\}$ is a vector space over $\mathbf{R}$. The vector addition and scalar multiplication are the usual addition and multiplication of complex numbers.

The next example is a generalization of Example 4. Although it appears rather trivial, it is of the utmost importance in the theory of fields.

EXAMPLE 5 Let E be a field and let F be a subfield of E. Then E is a vector space over F. The vector addition and scalar multiplication are the operations of E.

Subspaces

Of course, there is a natural analog of subgroup and subring.

Definition Subspace

Let V be a vector space over a field F and let U be a subset of V. We say that U is a *subspace* of V if U is also a vector space over F under the operations of V.

EXAMPLE 6 The set $\{a_2x^2 + a_1x + a_0 \mid a_0, a_1, a_2 \in \mathbf{R}\}$ is a subspace of the vector space of all polynomials with real coefficients over $\mathbf{R}$.

EXAMPLE 7 Let V be a vector space over F and let $v_1, v_2, \ldots, v_n$ be (not necessarily distinct) elements of V. Then the subset

$$\langle v_1, v_2, \ldots, v_n \rangle = \{a_1v_1 + a_2v_2 + \cdots + a_nv_n \mid a_1, a_2, \ldots, a_n \in F\}$$

is called the *subspace of V spanned by $v_1, v_2, \ldots, v_n$*. Any sum of the form $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ is called a *linear combination of* $v_1, v_2, \ldots, v_n$. If $\langle v_1, v_2, \ldots, v_n \rangle = V$, we say that $\{v_1, v_2, \ldots, v_n\}$ *spans V.* ∎

Linear Independence

The next definition is the heart of the theory.

Definition Linearly Dependent, Linearly Independent

A set S of vectors is said to be *linearly dependent* over the field F if there are vectors $v_1, v_2, \ldots, v_n$ from S and elements $a_1, a_2, \ldots, a_n$ from F, not all zero, such that $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$. A set of vectors that is not linearly dependent over F is called *linearly independent* over F.

In other words, a set of vectors is linearly dependent over F if there is a nontrivial linear combination of them over F equal to 0.

EXAMPLE 8 In $\mathbf{R}^3$ the vectors (1, 0, 0), (1, 0, 1), and (1, 1, 1) are linearly independent over $\mathbf{R}$. To verify this, assume that there are real numbers a, b, and c such that $a(1, 0, 0) + b(1, 0, 1) + c(1, 1, 1) = (0, 0, 0)$. Then $(a + b + c, c, b + c) = (0, 0, 0)$. From this we see that $a = b = c = 0$. ∎

Certain kinds of linearly independent sets play a crucial role in the theory of vector spaces.

Definition Basis

Let V be a vector space over F. A subset B of V is called a *basis* for V if B is linearly independent over F and every element of V is a linear combination of elements of B.

The motivation for this definition is twofold. First, if B is a basis for a vector space V, then every member of V is a unique linear combination of the elements of B (see Exercise 19). Second, with every vector space spanned by finitely many vectors, we can use the notion of basis to associate a unique integer that tells us much about the vector space. (In fact, this integer and the field completely determine the vector space up to isomorphism—see Exercise 31.)

EXAMPLE 9 The set $V = \left\{ \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} \middle| a, b \in \mathbf{R} \right\}$ is a vector space over $\mathbf{R}$ (see Exercise 17). We claim that the set $B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for V over $\mathbf{R}$. To prove that the set B is linearly independent, suppose that there are real numbers a and b such that

$$a\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives $\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so that $a = b = 0$. On the other hand, since every member of V has the form

$$\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

we see that B spans V. ■

We now come to the main result of this chapter.

Theorem 19.1 Invariance of Basis Size

If $\{u_1, u_2, \ldots, u_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ are both bases of a vector space V over a field F, then $m = n$.

PROOF Suppose that $m \neq n$. To be specific, let us say that $m < n$. Consider the set $\{w_1, u_1, u_2, \ldots, u_m\}$. Since the u's span V, we know that w_1 is a linear combination of the u's, say, $w_1 = a_1u_1 + a_2u_2 + \cdots + a_mu_m$, where the a's belong to F. Clearly, not all the a's are 0. For convenience, say $a_1 \neq 0$. Then $\{w_1, u_2, \ldots, u_m\}$ spans V (see Exercise 21). Next, consider the set $\{w_1, w_2, u_2, \ldots, u_m\}$. This time, w_2 is a linear combination of $w_1, u_2, \ldots, u_m$, say, $w_2 = b_1w_1 + b_2u_2 + \cdots + b_mu_m$, where the b's belong to F. Then at least one of $b_2, \ldots, b_m$ is nonzero, for otherwise the w's are not linearly independent. Let us say $b_2 \neq 0$. Then $w_1, w_2, u_3, \ldots, u_m$ span V. Continuing in this fashion, we see that $\{w_1, w_2, \ldots, w_m\}$ spans V. But then w_{m+1} is a linear combination of $w_1, w_2, \ldots, w_m$ and, therefore, the set $\{w_1, \ldots, w_n\}$ is not linearly independent. This contradiction finishes the proof. ■

Theorem 19.1 shows that any two finite bases for a vector space have the same size. Of course, not all vector spaces have finite bases. However, there is no vector space that has a finite basis and an infinite basis (see Exercise 25).

Definition Dimension
A vector space that has a basis consisting of n elements is said to have *dimension n*. For completeness, the trivial vector space {0} is said to be spanned by the empty set and to have dimension 0.

Although it requires a bit of set theory that is beyond the scope of this text, it can be shown that every vector space has a basis. A vector space that has a finite basis is called *finite dimensional*; otherwise, it is called *infinite dimensional.*

Exercises

Somebody who thinks logically is a nice contrast to the real world.

The Law of Thumb

1. Verify that each of the sets in Examples 1–4 satisfies the axioms for a vector space. Find a basis for each of the vector spaces in Examples 1–4.

2. (Subspace Test) Prove that a nonempty subset U of a vector space V over a field F is a subspace of V if, for every u and u' in U and every a in F, $u + u' \in U$ and $au \in U$. (In words, a nonempty set U is a subspace of V if it is closed under the two operations of V.)

3. Verify that the set in Example 6 is a subspace. Find a basis for this subspace. Is $\{x^2 + x + 1, x + 5, 3\}$ a basis?

4. Verify that the set $\langle v_1, v_2, \ldots, v_n\rangle$ defined in Example 7 is a subspace.

5. Determine whether or not the set $\{(2, -1, 0), (1, 2, 5), (7, -1, 5)\}$ is linearly independent over $\mathbf{R}$.

6. Determine whether or not the set

$$\left\{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$$

is linearly independent over Z_5.

7. If $\{u, v, w\}$ is a linearly independent subset of a vector space, show that $\{u, u + v, u + v + w\}$ is also linearly independent.

8. If $\{v_1, v_2, \ldots, v_n\}$ is a linearly dependent set of vectors, prove that one of these vectors is a linear combination of the other.

9. (Every spanning collection contains a basis.) If $\{v_1, v_2, \ldots, v_n\}$ spans a vector space V, prove that some subset of the v's is a basis for V.

10. (Every independent set is contained in a basis.) Let V be a finite-dimensional vector space and let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent subset of V. Show that there are vectors $w_1, w_2, \ldots, w_m$ such that $\{v_1, v_2, \ldots, v_n, w_1, \ldots, w_m\}$ is a basis for V.

11. If V is a vector space over F of dimension 5 and U and W are subspaces of V of dimension 3, prove that $U \cap W \neq \{0\}$. Generalize.

12. Show that the solution set to a system of equations of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\quad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0, \end{aligned}$$

where the a's are real, is a subspace of $\mathbf{R}^n$.

13. Let V be the set of all polynomials over Q of degree 2 together with the zero polynomial. Is V a vector space over Q?

14. Let $V = \mathbf{R}^3$ and $W = \{(a, b, c) \in V \mid a^2 + b^2 = c^2\}$. Is W a subspace of V? If so, what is its dimension?

15. Let $V = \mathbf{R}^3$ and $W = \{(a, b, c) \in V \mid a + b = c\}$. Is W a subspace of V? If so, what is its dimension?

16. Let $V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| \, a, b, c \in Q \right\}$. Prove that V is a vector space over Q, and find a basis for V over Q.

17. Verify that the set V in Example 9 is a vector space over $\mathbf{R}$.

18. Let $P = \{(a, b, c) \mid a, b, c \in \mathbf{R}, a = 2b + 3c\}$. Prove that P is a subspace of $\mathbf{R}^3$. Find a basis for P. Give a geometric description of P.

19. Let B be a subset of a vector space V. Show that B is a basis for V if and only if every member of V is a unique linear combination of the elements of B. (This exercise is referred to in this chapter and in Chapter 20.)

20. If U is a proper subspace of a finite-dimensional vector space V, show that the dimension of U is less than the dimension of V.

21. Referring to the proof of Theorem 19.1, prove that $\{w_1, u_2, \ldots, u_m\}$ spans V.

22. If V is a vector space of dimension n over the field Z_p, how many elements are in V?

23. Let $S = \{(a, b, c, d) \mid a, b, c, d \in \mathbf{R}, a = c, d = a + b\}$. Find a basis for S.

24. Let U and W be subspaces of a vector space V. Show that $U \cap W$ is a subspace of V and that $U + W = \{u + w \mid u \in U, w \in W\}$ is a subspace of V.

25. If a vector space has one basis that contains infinitely many elements, prove that every basis contains infinitely many elements. (This exercise is referred to in this chapter.)

26. Let $u = (2, 3, 1)$, $v = (1, 3, 0)$, and $w = (2, -3, 3)$. Since $(1/2)u - (2/3)v - (1/6)w = (0, 0, 0)$, can we conclude that the set $\{u, v, w\}$ is linearly dependent over Z_7?

27. Define the vector space analog of group homomorphism and ring homomorphism. Such a mapping is called a *linear transformation.* Define the vector space analog of group isomorphism and ring isomorphism.

28. Let T be a linear transformation from V to W. Prove that the image of V under T is a subspace of W.

29. Let T be a linear transformation of a vector space V. Prove that $\{v \in V \mid T(v) = 0\}$, the *kernel* of T, is a subspace of V.

30. Let T be a linear transformation of V onto W. If $\{v_1, v_2, \ldots, v_n\}$ spans V, show that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ spans W.

31. If V is a vector space over F of dimension n, prove that V is isomorphic as a vector space to $F^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in F\}$. (This exercise is referred to in this chapter.)

32. Show that it is impossible to find a basis for the vector space of $n \times n$ $(n > 1)$ matrices such that each pair of elements in the basis commutes under multiplication.

33. Let $P_n = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mid \text{each } a_i \text{ is a real number}\}$. Is it possible to have a basis for P_n such that every element of the basis has x as a factor?

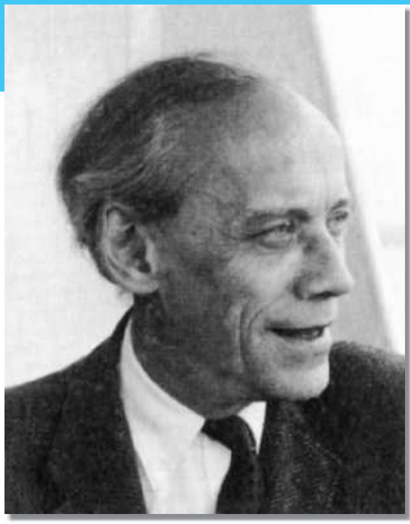
34. Find a basis for the vector space $\{f \in P_3 \mid f(0) = 0\}$. (See Exercise 33 for notation.)

35. Given that f is a polynomial of degree n in P_n, show that $\{f, f', f'', \ldots, f^{(n)}\}$ is a basis for P_n. ($f^{(k)}$ denotes the kth derivative of f.)

36. Prove that for a vector space V over a field that does not have characteristic 2, the hypothesis that V is commutative under addition is redundant.

37. Let V be a vector space over an infinite field. Prove that V is not the union of finitely many proper subspaces of V.

Emil Artin


Massachusetts Institute of Technology

For Artin, to be a mathematician meant to participate in a great common effort, to continue work begun thousands of years ago, to shed new light on old discoveries, to seek new ways to prepare the developments of the future. Whatever standards we use, he was a great mathematician.

RICHARD BRAUER,
Bulletin of the American Mathematical Society

EMIL ARTIN was one of the leading mathematicians of the 20th century and a major contributor to linear algebra and abstract algebra. Artin was born on March 3, 1898, in Vienna, Austria, and grew up in what was recently known as Czechoslovakia. He received a Ph.D. in 1921 from the University of Leipzig. Artin was a professor at the University of Hamburg from 1923 until he was barred from employment in Nazi Germany in 1937 because his wife had a Jewish grandparent. His family emigrated to the United States where he spent one year at Notre Dame then eight years at Indiana University. In 1946 he moved to Princeton, where he stayed until 1958. The last four years of his career were spent where it began, at Hamburg.

Artin's mathematics is both deep and broad. He made contributions to number theory, group theory, ring theory, field theory, Galois theory, geometric algebra, algebraic topology, and the theory of braids—a field he invented. Artin received the American Mathematical Society's Cole Prize in number theory, and he solved one of the 23 famous problems posed by the eminent mathematician David Hilbert in 1900.

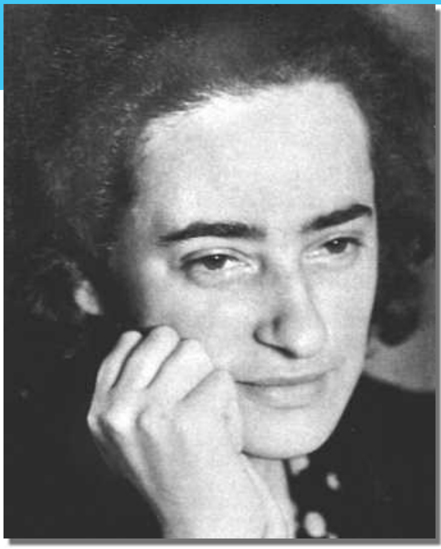
Eminent mathematician Hermann Weyl said of Artin "I look upon his early work in algebra and number theory as one of the few big mathematical events I have witnessed in my lifetime. A genius, aglow with the fire of ideas—that was the impression he gave in those years."

Artin was an outstanding teacher of mathematics at all levels, from freshman calculus to seminars for colleagues. Many of his Ph.D. students as well as his son Michael have become leading mathematicians. Through his research, teaching, and books, Artin exerted great influence among his contemporaries. He died of a heart attack, at the age of 64, in 1962.

For more information about Artin, visit:

http://www-groups.dcs.st-and.ac.uk/~history/

Olga Taussky-Todd



Courtesy of Edith H. Luchins, Professor of Mathematics and Adjunct Professor of Cognitive Science Emerita

"Olga Taussky-Todd was a distinguished and prolific mathematician who wrote about 300 papers."

EDITH LUCHINS AND MARY ANN McLOUGHLIN,
Notices of the American Mathematical Society, 1996

OLGA TAUSSKY-TODD was born on August 30, 1906, in Olmütz in the Austro-Hungarian Empire. Taussky-Todd received her doctoral degree in 1930 from the University of Vienna. In the early 1930s she was hired as an assistant at the University of Göttingen to edit books on the work of David Hilbert. She also edited lecture notes of Emil Artin and assisted Richard Courant. She spent 1934 and 1935 at Bryn Mawr and the next two years at Girton College in Cambridge, England. In 1937, she taught at the University of London. In 1947, she moved to the United States and took a job at the National Bureau of Standards' National Applied Mathematics Laboratory. In 1957, she became the first woman to teach at the California Institute of Technology as well as the first woman to receive tenure and a full professorship in mathematics, physics, or astronomy there. Thirteen Caltech Ph.D. students wrote their Ph.D. theses under her direction.

In addition to her influential contributions to linear algebra, Taussky-Todd did important work in number theory.

Taussky-Todd received many honors and awards. She was elected a Fellow of the American Association for the Advancement of Science and vice president of the American Mathematical Society. In 1990, Caltech established an instructorship named in her honor. Taussky-Todd died on October 7, 1995, at the age of 89.

For more information about Taussky-Todd, visit:

http://www-groups.dcs.st-and.ac.uk/~history

http://www.agnesscott.edu/lriddle/women/women.htm

CHAPTER 19 9TH EDITION
Solutions for odd-numbered exercises for the chapter on Vector Spaces

1. Each of the four sets is an Abelian group under addition. The verification of the four conditions involving scalar multiplication is straight forward. $\mathbf{R}^n$ has basis $\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,\ldots,1)\}$; $M_2(Q)$ has basis $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$;
   $Z_p[x]$ has basis $\{1,x,x^2,\ldots\}$; $\mathbf{C}$ has basis $\{1,i\}$.

3. $(a_2x^2+a_1x+a_0)+(a_2'x^2+a_1'x+a_0')=(a_2+a_2')x^2+(a_1+a_1')x+(a_0+a_0')$ and $a(a_2x^2+a_1x+a_0)=aa_2x^2+aa_1x+aa_0$. A basis is $\{1,x,x^2\}$. Yes, this set $\{x^2+x+1,x+5,3\}$ is a basis because $a(x^2+x+1)+b(x+5)=3c=0$ implies that $ax^2+(a+b)x+a+5b+3c=0$. So, $a=0, a+b=o$ and $a+5b+3c=0$. But the two conditions $a=0$ and $a+b=0$ imply $b=0$ and the three conditions $a=0, b=0$, and $a+5b+3c=0$ imply $c=0$.

5. They are linearly dependent, since $-3(2,-1,0)-(1,2,5)+(7,-1,5)=(0,0,0)$.

7. Suppose $au+b(u+v)+c(u+v+w)=0$. Then $(a+b+c)u+(b+c)v+cw=0$. Since $\{u,v,w\}$ are linearly independent, we obtain $c=0, b+c=0$, and $a+b+c=0$. So, $a=b=c=0$.

9. If the set is linearly independent, it is a basis. If not, then delete one of the vectors that is a linear combination of the others (see Exercise 8). This new set still spans V. Repeat this process until you obtain a linearly independent subset. This subset will still span V since you deleted only vectors that are linear combinations of the remaining ones.

11. Let u_1,u_2,u_3 be a basis for U and w_1,w_2,w_3 be a basis for W. Since dim $V=5$, there must be elements a_1,a_2,a_3,a_4,a_5,a_6 in F, not all 0, such that $a_1u_1+a_2u_2+a_3u_3+a_4w_1+a_5w_2+a_6w_3=0$. Then $a_1u_1+a_2u_2+a_3u_3=-a_4w_1-a_5w_2-a_6w_3$ belongs to $U\cap W$ and this element is not 0 because that would imply that

a_1, a_2, a_3, a_4, a_5 and a_6 are all 0.

In general, if $\dim U + \dim W > \dim V$, then $U \cap W \neq \{0\}$.

13. No. x^2 and $-x^2 + x$ belong to V but their sum does not.

15. Yes, W is a subspace. If (a, b, c) and (a', b', c') belong to W then $a+b = c$ and $a'+b' = c'$. Thus, $a+a'+b+b' = (a+b)+(a'+b') = c + c'$ so $(a, b, c) + (a' + b' + c')$ belongs to W and therefore W is closed addition. Also, if (a, b, c) belongs to W and d is a real number then $d(a, b, c) = (da, db, dc)$ and $ad + bd = cd$ so W is closed under scalar multiplication.

17. $\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} + \begin{bmatrix} a' & a'+b' \\ a'+b' & b' \end{bmatrix} = \begin{bmatrix} a+a' & a+b+a'+b' \\ a+b+a'+b' & b+b' \end{bmatrix}$
and $c\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} = \begin{bmatrix} ac & ac+bc \\ ac+bc & bc \end{bmatrix}$.

19. Suppose B is a basis. Then every member of V is some linear combination of elements of B. If $a_1v_1 + \cdots + a_nv_n = a_1'v_1 + \cdots + a_n'v_n$, where $v_i \in B$, then $(a_1 - a_1')v_1 + \cdots + (a_n - a_n')v_n = 0$ and $a_i - a_i' = 0$ for all i. Conversely, if every member of V is a unique linear combination of elements of B, certainly B spans V. Also, if $a_1v_1 + \cdots + a_nv_n = 0$, then $a_1v_1 + \cdots + a_nv_n = 0v_1 + \cdots + 0v_n$ and therefore $a_i = 0$ for all i.

21. Since $w_1 = a_1u_1 + a_2u_2 + \cdots + a_nu_n$ and $a_1 \neq 0$, we have $u_1 = a_1^{-1}(w_1 - a_2u_2 - \cdots - a_nu_n)$, and therefore $u_1 \in \langle w_1, u_2, \ldots, u_n\rangle$. Clearly, $u_2, \ldots, u_n \in \langle w_1, u_2, \ldots, u_n\rangle$. Hence every linear combination of $u_1, \ldots, u_n$ is in $\langle w_1, u_2, \ldots, u_n\rangle$.

23. Since $(a, b, c, d) = (a, b, a, a + b) = a(1, 0, 1, 1) + b(0, 1, 0, 1)$ and $(1, 0, 1, 1)$ and $(0, 1, 0, 1)$ are linearly independent, these two vectors are a basis.

25. Suppose that $B_1 = \{u_1, u_2, \ldots, u_n\}$ is a finite basis for V and B_2 is an infinite basis for V. Let $w_1, w_2, \ldots, w_{n+1}$ be distinct elements of B_2. Then, as in the proof of Theorem 19.1, the set $\{w_1, w_2, \ldots, w_n\}$ spans V. This means that w_{n+1} is a linear combination of $w_1, w_2, \ldots, w_n$. But then B_2 is not a linearly independent set.

27. If V and W are vector spaces over F, then the mapping must preserve addition and scalar multiplication. That is, $T : V \to W$ must satisfy $T(u+v) = T(u) + T(v)$ for all vectors u and v in V, and $T(au) = aT(u)$ for all vectors u in V and all scalars a in F. A vector space isomorphism from V to W is a one-to-one linear transformation from V onto W.

29. Suppose v and u belong to the kernel and a is a scalar. Then $T(v+u) = T(v) + T(u) = 0 + 0 = 0$ and $T(av) = aT(u) = a \cdot 0 = 0$.

31. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. The mapping given by $\phi(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = (a_1, a_2, \ldots, a_n)$ is a vector space isomorphism. By observation, ϕ is onto. ϕ is one-to-one because $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ implies that $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$. Since $\phi((a_1v_1+a_2v_2+\cdots+a_nv_n)+(b_1v_1+b_2v_2+\cdots+b_nv_n)) = \phi((a_1+b_1)v_1 + (a_2+b_2)v_2 + \cdots + (a_n+b_n)v_n) = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n) = (a_1, a_2, \ldots, a_n)+(b_1, b_2, \ldots, b_n) = \phi(a_1v_1 + a_2v_2 + \cdots + a_nv_n) + \phi(b_1v_1 + b_2v_2 + \cdots + b_nv_n)$ we have shown that ϕ preserves addition. Moreover, for any c in F we have $\phi(c(a_1v_1 + a_2v_2 + \cdots + a_nv_n)) = \phi(ca_1v_1 + ca_2v_2 + \cdots + ca_nv_n) = (ca_1, ca_2, \ldots, ca_n) = c(a_1, a_2, \ldots, a_n) = c\phi(a_1v_1 + a_2v_2 + \cdots + a_nv_n)$ so that ϕ also preserves scalar multiplication.

33. No, for 1 is not in the span of such a set.

35. Write $a_1f + a_2f' + \cdots + a_nf^{(n)} = 0$ and take the derivative n times to get $a_1 = 0$. Similarly, get all other a_i's $= 0$. So, the set is linearly independent and has the same dimension as P_n.

37. Suppose that $V = \bigcup_{i=1}^n V_i$ where n is minimal and F is the field. Then no V_i is the union of the other V_j's for otherwise n is not minimal. Pick $v_1 \in V_1$ so that $v_1 \notin V_j$ for all $j \neq 1$. Pick $v_2 \in V_2$ so that $v_2 \notin V_j$ for all $j \neq 2$. Consider the infinite set $L = \{v_1 + av_2 \mid a \in F\}$. We claim that each member of L is contained in at most one V_i. To verify this suppose both $u = v_1 + av_2$ and $w = v_1 + bv_2$ belong to some V_i. Then $u - w = (a-b)v_2 \in V_i \cup V_2$. By the way that v_2 was chosen this implies that $i = 2$. Also, $bu - aw = (b-a)v_1 \in V_i \cup V_1$, which

implies that $i = 1$. This contradiction establishes the claim. Finally, since each member of L belongs to at most one V_i, the union of the V_i has at most n elements of L. But the union of the V_i is V and V contains L.