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Research with Pre-Mathematicians

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1. Introduction

The purpose of these remarks is to record the author's experience with and resulting thoughts about research with "pre-mathematicians." By this I mean young (generally pre-graduate school) people who have the ability to enter a career in high-level mathematics. Since such a career will require substantial, independent, creative mathematical thought, an honest and serious acquaintance with mathematical research is a valuable complement to conventional course work. Fortunately, a taste of research can help to seduce bright people into mathematics. The appeal of mystery and empowerment given by success in research can be most addictive.

Much of my experience in research with pre-mathematicians has been via NSF support, both individual grants and site grants for summer REU programs, since 1989. A brief history and account of that activity may be found in [**1**].

The remainder of these remarks is organized into topical sections, beginning with a short update on the REU program at W&M. Then, we discuss where the most essential ingredient, the problems, come from, the value of undergraduate research, and some examples of recent research. In a future piece, I hope to record the talk I give "Doing Research in Mathematics" to REU students. It is designed to prepare them for the inevitable frustration inherent in serious mathematical research.

2. Update on REU at W&M

Our concept of summer REU at W&M has always been collaborative research in small groups (at least one pre-mathematician and at least one mentor, no more than 4 total) on serious research problems whose outcome will be of interest to others. The unifying theme has been matrix analysis and applications, often with a considerable combinatorial flavor. It has proven beneficial that students, not working together, can chat and appreciate problems in the same general area. The collection of mentors has never been the same from one summer to the next. Several faculty members have gradually drifted away from the activity. It became clear that some were not very good at it, while others are not so committed to students or had difficulty committing the necessary block of time in view of other activities. I am now the only one who has done it every summer. We have often had visitors help as

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mentors. These range from senior and junior collaborators to postdocs, other junior visitors, and Ph.D. students who are "graduates" of the REU program. This has worked very well for all parties (and has been very nice for me to have collaborators around for a sustained period). For example, in 2005, I had 3 junior colleagues from Poland come for the summer (two visiting the US for the first time).

The supported students have been admitted (primarily) competitively from a national applicant pool that is generally around 160. A very large percentage of these would benefit from an REU and be able to do good work. We try to achieve some gender balance, which is easier in some summers than others. When female candidates have declined our offers, it is usually to choose a topic more to their liking or to attend a program dedicated to females. It is important to attract able females to mathematics and to give them a realistic view of mathematical research (just as it is to attract talent from any other quarter). It may be that the time for female-dedicated programs has past. The only other consideration sometimes taken into account in admission is that a student who seems very strong, but has not had the opportunity for exposure to serious mathematics in their undergraduate experience thus far, may be given extra consideration. This entails some risk, but can also be very beneficial to the student, meeting an REU objective.

In addition, we have frequently had students not supported by the NSF involved in the program; these range from high school students to foreign nationals (from Portugal, Ireland, Korea, etc.) to qualified "walk-ons". The resulting vertical integration (e.g. a high school student, working with an undergraduate, working with a graduate student, working with me) has worked well and been beneficial to all parties. In fact, I strongly feel that carefully selected high school students should be involved in REU, and that, at rather low cost, this would serve all goals of the REU concept. In a like manner cooperative agreements with foreign countries could be helpful. Ireland is already implementing an REU program that would allow participation by foreign students. It would be of value if, at least, cooperative programs with other countries would allow US students to be involved in REU-like research abroad, while students from those countries came here.

3. Were do Problems Come From?

Perhaps the most important feature of an activity designed to expose pre-mathematicians to research is the availability of appropriate problems. If the activity is of some significant scale, such as a recurring summer program involving several students each summer, recurring availability of many good problems is both essential and difficult. Good problems for this purpose should be (*i*) accessible, (*ii*) unsolved, and (*iii*) important; (*iv*) it should be likely that some valuable progress can be made, but (*v*) the area should be large and open-ended enough that an exhaustively complete solution is unlikely before the end of a summer program. Importance is a judgment but the problem area should relate to other things or be an interesting part of a bigger picture and should be viewed as intriguing by the pre-mathematicians. We have had many large problems progress in parts over many summers. Accessibility depends, to some extent, on participant background.

Our REU activity has been centered around matrix analysis/linear algebra and its applications, broadly defined. A solid beginning linear algebra course is required for admission and a true second course is preferred. Each participant is given a copy of [**2**] at the beginning of the program, and copies of [**3**] are readily available. The participants are rapid learners, and learning that helps solve their problem is a very compelling motivation. With on-line and library resources and guidance from someone who best knows the area, the topic of matrix analysis has proved ideal as a problem source. (In addition, because of its connections with all parts of mathematics and most of its applications, learning about the area serves the participants well.)

As should be the case with any guided research activity, the ultimate source of problems is spin-off from an active research program of a top researcher. In my case, I have very broad interests in matrix analysis and combinatorics and their relationship to other parts of mathematics. I collaborate with several dozen mathematicians in the US and around the world. This naturally suggests many related problems and subproblems that can be very useful to the collaborations. In addition, I frequently receive e-mail queries, some of which the writer would like help with or suggest yet other questions. Occasionally, good problems are simply suggested by colleagues. Since prior REU work and ongoing work raise many fresh questions as well, this altogether allows accumulation in a year's time of many "good" problems, many more than I can pursue by myself.

Over the years, we have had many continuing themes for REU problems. Each has been a very rich source of specific problems and nice results, many of which have been published. These include: (*i*) matrix completion problems; (*ii*) the long standing conjecture from statistical physics that the coefficients of $p(t) = Tr[(A+tB)^m]$ are positive whenever A and B are positive definite matrices; (*iii*) possible multiplicities of the eigenvalues among the Hermitian matrices with a given graph; (*iv*) minimum rank among positive semidefinite matrices with a given graph; and (*v*) factorization of matrix and operator functions.

4. The Benefits of Undergraduate Research/REU

If the purpose of REU programs (in mathematics) is to attract strong students to mathematics by giving them a realistic view of mathematical research, that seems to be working well. (But, recent statistics about a return to low percentages of US students among those finishing Ph.D.'s in the US are a cause for concern.) To be sure, many successful REU students would have gone on to mathematics Ph.D. study anyway, but it is likely that there are many others who would not have gotten "into" mathematics were it not for the summer opportunities. However, any program has consequences beyond those intended and I want to mention here some benefits that I perceive besides those intended or often cited.

(i). It was for many years a wonderful tradition in Russia and parts of Eastern Europe that important and established mathematicians would go out of their way to nurture talented pre-mathematicians. This tradition not only improved the discipline and helped new entrants feel a part of the "community," but it also helped establish famous mathematical traditions and establish mathematics as an enduring cultural tradition that transcended politics that came and went. As a symbol, the commitment to REU and research with pre-mathematicians is a modern version of that tradition that helps to accomplish similar objectives here.

(ii). The actual research that results from REUs and the like should not be ignored. In fact it seems to me one of the most important tangible products of the activity. In my experience, I am able to pursue interesting questions that I would not have been able to otherwise and, though I have altruistic motives as well, I would

not be as enthusiastic were it not for the actual results. (As a mathematician I am rather social and enjoy working with others as well.) It also provides a nice way to establish pieces of work that can be assembled into a bigger picture. See the last section for some examples. In many cases, the actual work done is important and is published in very credible journals, and we should be assembling and publicizing the examples. The research is also very inexpensive to the NSF.

(iii). What may seem to be a failure may also be a success. If a student finds through exposure to research, that it really wasn't for him, that is much better (and cheaper) than finding it out from 2 years of graduate school. Fortunately, this is an unusual outcome.

(iv). The reinforcement of interest in and commitment to mathematics that comes from a group working together (even if not on the same problem) should not be underestimated. I had not anticipated it, but realized very early that one of the biggest benefits to the students (especially those who were "one-of-a-kind" at their institutions) is being with others of similar interest and outlook. This may result in friendships and acquaintances that persist through a career, and in fact I have long term collaborators who were once my REU students. We also have an example of a marriage of two REU students who are now professors at Bucknell!

Let me indulge in closing this section with a story of my first pre-mathematician research experience as a mentor. (I had none as a student, though I did publish a paper without a mentor.) I was an NRC-NAS postdoc at the National Bureau of Standards (now NIST), which, at the time, had an Eastern European-like tradition of hiring Westinghouse talent research winners in the summer. I ended up working with one, Tom Leighton, which resulted in two very nice papers, a classic on possible sign patterns of inverse positive matrices and a very practical one on graph isomorphism and eigenvalues. Tom did well academically and ended up founding Akamei Corporation (along with a junior colleague, who, by chance, had been a student with my collaborator Raphy Loewy, and who tragically died on 9-11-01), which survived the bursting of the "tech bubble" and is now a very successful company. The value added to the economy by Akamei alone likely dwarfs the low cost of REU programs (and I wish that I had gotten agreement from REU-type students for only, say 0.1% of their life-time earnings!)

5. Some Examples of Recent Research

I have now published about 40 of my well over 300 papers with pre-mathematicians (many papers involving several co-authors), and quite a few others are in some stage of progress. In addition, there is a comparable total amount of publication by colleagues with undergraduates. In my case, I would not have had time to pursue much of this work were it not for the interaction with students. Yet, I am extremely happy with and proud of much of this work, which has demonstrably advanced the subject.

We mention a few specific examples here, but, for variety, do not include examples from the continuing themes already mentioned in Section 3. This should help to make clearer some comments made in other sections. Nothing else guides the choice of these few examples.

5.1 An n-by-n matrix A over a field has an LU factorization if there exist a lower triangular matrix L and an upper triangular matrix U (wlog over the same field) such that $A = LU$. Such a factorization is important both in theory and in many applications and is calculated, under certain circumstances, as early as in the first course in linear algebra. Though some sufficient and some necessary conditions are known for its existence, I realized that no characterization of matrices A for which an LU factorization exists was known. It definitely need not exist, and the sufficient conditions were not generally necessary, while the necessary conditions were not sufficient. In the REU program one summer Pavel Okunev took this up with me. I had noticed that it was necessary that the rank of the upper left k-by-k principal submatrix of A plus k should be at least the rank of the first k rows of A plus the rank of the first k columns.

We wondered if this were sufficient, which would require understanding how to arrive at an L and a U. Eventually, Pavel came up with a complicated proof of sufficiency which worked for an algebraically closed field. But, we felt that the answer should not depend upon the field. Finally, we found a field independent proof of sufficiency, one of the more fundamental results of a summer program and the first ever to reach its original objective before the end (a risk in a narrowly focused problem). But, there is always more to do. The answer raised the question, for a matrix A without an LU factorization, of how many and where entries above the diagonal of L and/or below the diagonal of U might be needed for a factorization. We gave partial answers, and fuller answers have been given with Maribel Bueno.

5.2 The natural partial order on n-by-n Hermitian matrices A and B is the positive semidefinite partial order: $A \geq B$ iff $A - B$ is PSD. It has long been known that if A and B are positive (semi-)definite, then $A \geq B$ implies $A^t \geq B^t$ for all $0 \leq t \leq 1$. In a conversation with a Polish statistician, Czeslaw Stepniak, the question arose of for which pairs of positive (semi-)definite matrices A and B should we have $A^t \geq B^t$ for all $t \geq 1$? It is clearly necessary that $A \geq B$, and this is sufficient if A and B commute. Another obvious sufficient condition is that all eigenvalues of A are at least any eigenvalue of B. It was not clear what a characterization should be, but in summer 2006, I took this up with undergraduate Becky Hoai and interested colleague, Ilya Spitkovsky (who often advises REU students). Another student also had an interest. After some thought we came up with and proved two characterizations, one very pretty and one of likely practical value. The former is that k matrices $C_1 \geq ... \geq C_k$ that pair-wise commute (i.e. C_i communites with $C_{i+1}, i = 1, ..., k-1$ and A commutes with C_1 and B with C_k) may be inserted between A and B: $A \geq C_1 \geq ... \geq C_k \geq B$. The number of matrices k may have to be as high as $n-1$, but not higher. Several related results were given.

5.3 If the zero/nonzero pattern A of an m-by-n matrix over a field is known, but not the values of the nonzero entries, the question of what the rank might be often arises. The maximum possible rank has long been understood, and all ranks between the minimum and maximum occur, but the minimum is quite difficult to characterize. If there is a k-by-k subpattern of A (in general position) that is permutation equivalent to a triangular pattern with nonzero diagonal (call this a "k-triangle") then the min rank of $A \geq k$. Further, min rank $A \geq T(A)$, the maximum of such k or the "triangle size" of A.

It was known from prior work of the author that min rank $A > T(A)$ can occur, the smallest known example being 7-by-7. This raises a natural question of for which m, n, r must an m-by-n pattern of minimum rank r (over the real field) have an r-triangle? Rafael Cauto (Spain) and I had obtained some important partial results about this, and I had suggested the question to many bright pre-mathematicians who had given up after a brief look. In 2005, undergraduate Josh Link took up the question quite seriously, initially with a combination of clever, ad hoc arguments and computing. Eventually, he and I found a clever (but still involved) complete solution, which I would never have found without his collaboration. This settled a long-standing natural question, but raises many more. For which m, n, r is the "first" instance of an m-by-n pattern A for which min rank $A = r = T(A) + 2$?

5.4 A square matrix is Toeplitz if its entries are constant along diagonals parallel to the main one. As a natural case in a progression of determinantal inequality questions, we raised the following question in summer 2006: which ratios of products of principal minors are bounded among all positive definite Toeplitz matrices? Two pre-mathematicians, Hyo-min Choi from Korea and Alex Porush, a bright high school student from the area, took up this problem. I had addressed such questions before for M-matrices (and inverse M-matrices), totally positive matrices, positive definite matrices and certain structured P-matrices with co-authors Shaun Fallat and Tracy Hall (ex REU). The general positive definite case, especially, is still very unresolved (important partial results) and presents some remarkable difficulties, but a cone theoretic approach evolved from that work. Choi and Porush used this approach but had much creative work to do. Positive semi-definite Toeplitz matrices with special distributions of rank among the principal submatrices had to be constructed (or their existence ruled out) and inequalities suggested by the method had to be proven. They pushed things through $n = 6$, a remarkable and fresh piece of work in a very classial area.

5.5 Ron Smith and I had pioneered linear interpolation problems for special classes of matrices. Example: for which pairs x and y of real n-vectors does there exist a P-matrix (positive principal minors) A such that $Ax = y$. We had found informative characterizations for many familiar classes.

This raised a natural (further) question, what about replacing x, y by n-by-k matrices X and Y (wlog of full rank k). This proves to be enormously challenging for virtually every class. I had done the positive definite case, which had a nice answer, but we hadn't done any other. We were concentrating on the P-matrix case and had a natural conjecture, but could not even prove it in the case $n = 3, k = 2$. Christian Sykes took up the problem and focused upon the 3,2 case. This is very geometric and analytic, which he liked. Ron and I had been trying to find an elegant 3,2 proof that might generalize. I encouraged Christian not to worry about how many cases he might have to consider, but that resolution of the 3,2 case would affect everyone's thinking. In the end, he found a rather nice algebraic proof of our conjecture via many cases. He has already spoken about this at a meeting in Portugal and won a prize for the talk.

References

[1] C.R. Johnson and D.J. Lutzer, A Decade of REU at William and Mary, pp.19-29, in Proceedings of the Conference on Summer Undergraduate Research Programs, J. Gallian, ed, American Mathematical Society, Providence, 2000.
[2] R. Horn and C.R. Johnson, Matrix Analysis, Cambridge University Press, NY 1985.
[3] R. Horn and C.R. Johnson, Topics in Matrix Analysis, CUP, NY, 1990.

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