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LSU REU: Graphs, Knots & Dessins in Topology, Number Theory & Geometry

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1. Overview

The LSU Math REU has run continuously since 1993, with funding from both the Louisiana Board of Regents and from the National Science Foundation. The targeted participants will normally be applying for their first REU experience and will have done well in an undergraduate abstract algebra course. A diversely balanced group of participants, especially in terms of gender balance and type of home institution (university versus four-year college) is selected from the national pool of applicants.

Our mathematical focus *Graphs, Knots & Dessins* lies in the overlap of Geometry, Topology, and Number Theory where many fertile interconnections can be explored. We seek out problems of current research interest that can be approached with the tools of group actions, group presentations, group representations, and various types of counting functions (such as a zeta function) where explicit computations can be made by undergraduates. We make a special effort to choose sets of problems that center around a common technique or theme so that students can interact with each other.

The Mathematics REU at LSU has these goals: to involve the next generation of young people in high level mathematical research in areas that are currently active, and to foster clear communication of mathematical results.

The basic structure of the REU is this: After a week of lectures by the mentors providing background, motivation, examples, general research directions, and specific entry points, each student selects a project on which to work. A cluster is formed consisting of a faculty mentor, a graduate assistant, and four students working on related problems. Each student then works primarily within the cluster and directly with the mentor for the rest of the summer. However, the entire group (twelve students and faculty and graduate mentors) meet each afternoon for mathematical conversations, refreshments and announcements relevant to the entire group. The clusters help develop strong collegial student-mentor and student-student relations, and the daily larger meetings help build a group identity.

On week-ends there are parties for the students at faculty homes, or field trips to nearby sites of interest. When the participants leave the program they should view

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Louisiana and LSU as intellectually stimulating places, they will have made some life-long friendships, and they will have made meaningful progress on challenging problems. As a result, many will decide to pursue graduate study in mathematics.

The students give three oral presentations of their work: a ten-minute statement of their chosen problem in the second week, a fifteen-minute progress report in the middle, and a twenty-minute final project summary at the end. Each student turns in a written project report, prepared in LaTeX in the form of a research paper.

2. Nature of Student Activity

The student projects center around problems in graphs, knots and *dessins d'enfants* in topology, number theory and geometry. This quickly leads to sophisticated mathematics. Nonetheless there are entry points suitable for getting students to work successfully in these areas. Often this involves computation and judicious construction of examples.

Significant features of the program include:

- The PIs lecture during the first week on interesting project directions, focusing on accessible entry points. Sometimes a professor from outside LSU is invited to present material. We believe that it is important that students have some idea of the larger context in which their chosen problem sits.
- The students select their own problems on which to work within the areas proposed in the lectures. In the next days, each student gives a short lecture explaining the problem in simple terms. This process gives the student a sense of commitment, ownership, and motivation.
- Student research starts with examples and computation. By the end of the program students formulate conjectures and prove theorems.
- Students work within clusters. They have individual projects, but interact with other participants. There are clusters of related projects and the students in a cluster are encouraged to help each other.
- Interaction with graduate students. This component has been possible through additional funding from the Louisiana Board of Regents.
- Communication of the results through oral and written presentation. Final projects are written in LaTeX and put on the REU website. The students give three oral presentations: a short project description at the beginning, a midterm progress report and a final report.
- Publication in undergraduate journals, and in regular mathematics research journals when appropriate. We encourage and financially support former REU participants to attend AMS poster sessions and other math conferences where the students can present their research.

We have also featured lectures by visiting experts. In 2002 Kiyoshi Igusa came from Brandeis to give a series of lectures on *pictures* related to free resolutions of modules over certain kinds of groups, and in 2006 Sergei Chmutov came from Ohio State to give lectures on knot and link invariants. In the weeks after Chmutov's lectures there were two videoconferences between the Ohio State University VIGRE and the LSU Math REU with talks by students at each institution.

3. Research Environment

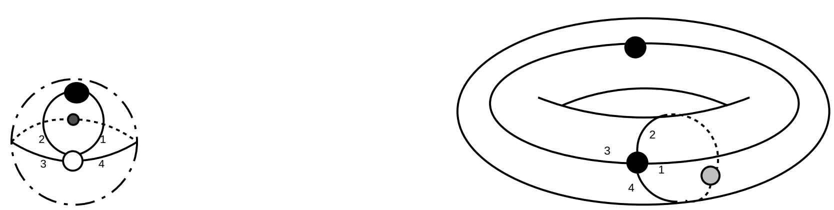
Our projects center around our theme of *Graphs, Knots & Dessins in Topology, Number Theory & Geometry.* This continues the general direction of our past summer REUs (low dimensional topology, graphs, zeta functions), but with a new emphasis, that of *dessin*, a combinatorial objects which, informally, is a graph together with a cyclic ordering of the edges meeting at a vertex. This additional information provides an embedding of the graph into an oriented surface. There are other names for this concept (ribbon graph, fat graph, *dessins d'enfants*) and they occur in multiple, diverse contexts. We will use the term *dessin.*

Twenty years ago Grothendieck (who coined the term *dessins d'enfants*) discovered unexpected connections to algebraic geometry and Galois theory. In fact, the absolute Galois group of the rational field $\mathbb{Q}$ acts faithfully on the set of dessins, but this action is far from understood. The action of Galois groups on the fundamental group of $\mathbf{P}^1 - \{0, 1, \infty\}$ has led to profound studies and conjectures (by Deligne, Drinfeld, Goncharov, Ihara and others), some of which can be attacked by advanced undergraduates. This is an exciting domain whose problems will not be exhausted for many years.

What makes dessins especially suitable for our REU is that there are entry points that are completely elementary. Start with an ordered pair (σ_0, σ_1) of permutations in a symmetric group S_n. This is all the information required for the specification of a *bipartite dessin.* The set of edges is $E = \{1, 2, \ldots, n\}$, the bipartition of the vertices is given by the orbits of the two permutations and the endpoints of an edge are the orbits in which it lies. The faces of the dessin are the orbits of σ_2 determined by the identity: $\sigma_0\sigma_1\sigma_2 = \mathrm{Id}$. This determines a bipartite graph (which we picture with hollow vertices for the orbits of σ_0 and dark vertices for the orbits of σ_1. Attaching k-gons to the edge cycle determined by the k-cycles of σ_2, we construct an (isotopy class of an) embedding of the graph into a surface. The genus of the surface is determined by the usual Euler characteristic formula, $2 - 2g = v - e + f$, where v is the total number of vertices; e, the number of edges and f, the number of faces, respectively.

For an example, let $n = 4$ and consider the genus 0 dessin $\mathbb{D}_1 = (\sigma_0 = (1234), \sigma_1 = (12)(34))$ and the genus one dessin $\mathbb{D}_2 = (\sigma_0 = (1234),\ \sigma_1 = (13)(24))$. The dessins are:

FIGURE 1. Two Dessins



The following theorem concerning bipartite dessins summarizes work of Grothendieck and Belyi:

THEOREM (Grothendieck-Belyi). *The following are in a canonical one to one correspondence:*

(1) *Isomorphism classes of ordered pairs* (σ_0, σ_1) *where the group* $\langle \sigma_0, \sigma_1 \rangle$ *is a transitive subgroup of* S_n*.*
(2) *Connected graphs* Γ *with a bicoloring of its vertices, and for each vertex of the graph, a fixed cyclic (counterclockwise) order of the edges incident on that vertex.*
(3) *Isotopy classes of* (X, Γ) *where* X *is a compact oriented surface,* $\Gamma \subset X$ *is a connected graph, with a bicoloring of its vertices, and such that* $X - \Gamma$ *is a union of contractible* 2*-cells.*
(4) *Isomorphism classes of* (X, β) *where* X *is a nonsingular irreducible projective algebraic curve defined over an algebraic number field, and* $\beta : X \to \mathbf{P}^1$ *is a morphism, also defined over a number field, such that* β *ramifies only over* $0, 1, \infty \in \mathbf{P}^1$*.*

Moreover, every nonsingular irreducible projective algebraic curve defined over a number field admits a morphism β *as in the fourth item above.*

The following gives a selection of problems related to dessins suitable for REU students.

Gassmann triples: A Gassmann triple (G, H, H') of groups together with a choice of two elements $g_0, g_1 \in G$ can be used to create two different pairs of graphs: Their dessins, and their Cayley-Schreier graphs. For the dessins, the coset spaces $G/H, G/H'$ label edges while for the Cayley-Schreier graphs the cosets spaces label vertices. Find an algorithm to relate the dessins to the Cayley-Schreier graphs. Try to use that algorithm to relate the Ihara zeta function of the graph underlying a dessin to the Ihara zeta function of the Cayley-Schreier graph.

Discrete Jacobians of Dessins: For each dessin $\mathbb{D}$, Shabat and Voevodsky, [**SV90**], define an abelian variety $J_{\mathbb{D}}$ which we will call the *discrete Jacobian* of the dessin. Their fundamental result is that, under a subdivision process, these Jacobians converge in a suitable sense to the usual Jacobian of the Riemann surface X in which the dessin is embedded, [**SV89**], [**Cav99**]. In contrast to the Jacobian of a Riemann surface, whose definition involves computation of periods of integrals, the discrete Jacobians of dessins have an elementary combinatorial definition, and so far have not been studied in detail. Investigate these discrete Jacobians.

Bollabas-Riordan-Tutte polynomial in link theory: Some of these problems involve an important polynomial invariant of dessins, called the BRT polynomial. It was defined by Bollobas and Riordan [**BR02**] in a manner generalizing the spanning tree construction of the Tutte polynomial of a graph. Very recently, new connections between the theory of dessins and knot/link theory was discovered by two groups Chmutov & Pak [**CP**], and Dasbach, Futer, Kalfagianni, Lin & Stoltzfus [**DFK$^+$**], by relating the BRT polynomial of certain dessins constructed from link diagrams to the Jones polynomial of the link. Find a formula for the BRT polynomial of the 2-parallel (more generally: n-parallel) of a knot in terms of the original knot. In particular, develop an understanding of this construction from the relationships among the permutations of the associated dessins.

Past graduates of the LSU REU program continue to make mathematical news. Shelley Harvey, now Assistant Professor at Rice University has just received a Sloan Fellowship. Her description of her experience at the LSU (and Cornell) REU was published in the Notices of the AMS [**Har98**]. The semester before participating in our REU she was majoring in engineering. A second graduate, Dorothy Buck, of the Imperial College, London, will give an MAA Invited Address at MathFest this year, August 10-12, in Knoxville, TN. She is an expert in the developing field of mathematical biology. Her talk is entitled "The Circle (and Knot and Link) of Life: How Topology Untangles Knotty DNA Questions".

We also had three of our past participants, Yaim Cooper (2005), Michele Lastrina (2005) and Stacey Goff (2003), participate in the 2006 Women and Mathematics Program of the Institute for Advanced Study and Princeton University entitled *Zeta functions all the way*. Michele and Yaim also presented posters in the AMS 2006 National meeting in San Antonio. Yaim is preparing a paper, *Properties determined by the Ihara zeta function of a graph*, for publication. Selected additional publications are listed in the references.

4. Student Recruitment and Selection

Participants are recruited nationally, and are drawn from all major regions of the country. Over the years we have worked to recruit students from underrepresented groups into our program. In particular we have been successful in achieving gender balance. We also strive to provide places for students from four-year colleges in addition to those we recruit from major universities.

From the participants during the years 2001-2004, 33 (out of 48) subsequently attended graduate school. For the same years, 18 of the participants were women and 16 were from four-year college programs.

The initial contact with applicants has been through recommendation from previous participants, announcements in professional notices, or via the Internet. Our web page (`http://www.math.lsu.edu/REU.html`) includes detailed information on the program and provides an email address and electronic application forms. We also request that links to our web page are present at mathematics portal sites throughout the Internet.

We receive between one and two hundred email inquiries about our program each year. Each inquiry generates an electronic information pack that includes a description of the program. We request that applicants supply an academic record supplemented by descriptions of advanced courses they have taken, two letters of reference, and a personal statement explaining their interest in the REU.

Our website is set up so that applications can be made completely electronically, except for official transcripts. Reference letters can be emailed. We select students based on evidence of their ability in mathematics, willingness to commit to completing a project, and ability to interact well with others. We normally require a participant in our program to have successfully completed a course in abstract algebra; this is necessary for working at the mathematical level of our program. Many students come to the program with advanced courses such as algebraic topology, or algebraic geometry. Finally, we give preference to students seeking their first REU experience.

References

[BR02] Béla Bollobás and Oliver Riordan, *A polynomial of graphs on surfaces*, Math. Ann. **323** (2002), no. 1, 81–96. MR **MR1906909 (2003b:**05052)

[Cav99] Renzo Cavalieri, *Curve algebriche definite su* $\mathbb{Q}$, Master's thesis, Universit'a degli studi di Milano, 1999, http://www.math.lsa.umich.edu/˜ crenzo/professional.html.

[CP] Sergei Chmutov and Igor Pak, *The Kauffman bracket and the Bollobas-Riordan polynomial of ribbon graphs*, arXiv:math.GT/0404475.

[Cza05] Debra Czarneski, *Zeta functions of finite graphs*, Ph.D. thesis, Louisiana State University, 2005, http://etd.lsu.edu/docs/available/etd-07072005-121013/.

[DFK+] Oliver T. Dasbach, David Futer, Efstratia Kalfagianni, Xiao-Song Lin, and Neal W. Stoltzfus, *The Jones polynomial and dessins d'enfant*, arXiv:math.GT/0605571.

[GS] Josh Genauer and Neal W. Stoltzfus, *Explicit diagonalization of the Markov form on the Temperley-Lieb algebra*, arXiv:math.QA/0511003, accepted for publication in Math. Proc. Cambridge Phil. Soc.

[Har98] Andrea Harvey, Shelley & Ritter, *A research experience for undergraduates*, Notices of the American Mathematical Society (1998), no. 2, 267–68.

[Kha06] Zuhair Khandker, *Defining a zeta function for cell products of graphs*, R. H. U.G. Math. Journal **7** (2006), no. 1, http://www.rose-hulman.edu/mathjournal/.

[Pra06] Neeraj Pradhan, *Hyperplane arrangements and the Bernstein-Gelfand-Gelfand correspondence*, CUSJ **1** (2006), no. 1, 28–34, http://cusj.columbia.edu/.

[SV89] G. B. Shabat and V. A. Voevodsky, *Piecewise Euclidean approximations of Jacobians of algebraic curves*, preprint, 1989.

[SV90] ———, *Drawing curves over number fields*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 199–227. MR **MR1106916 (92f:**11083)

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