Proceedings of the Conference on
Promoting Undergraduate Research in Mathematics

My Experiences Researching With Undergraduate Mathematicians: The Collaboration Model

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During the past six years as a Visiting Assistant Professor at Kenyon College, Colorado College, and St. Olaf College, I collaborated with 15 undergraduate students on 11 research projects, both singly and in groups of two or three. I say "collaborated," rather than "supervised," since my approach to undergraduate research is to model my work with students on my collaborations with other Ph.D. mathematicians. In my experiences speaking with participants at the Conference on Promoting Undergraduate Research in Mathematics, it seems that this approach is unusual enough that others might benefit from a description of my experiences, ideas, and results. This article summarizes what I believe to be the most significantly different aspects of my approach, and what I feel were the most important ingredients of my successful research experiences.

1. Finding a topic

There are two criteria that I look for when I select a research problem for collaboration with students.

The problem is new to me and to the student. In order to be able to collaborate on a problem as equal contributors, it's necessary that my students and I start at roughly the same point in the research, in knowledge of the relevant background information, and experience with techniques in the area. As a rule, I try not to work on problems with students that I have previously studied and published results on. An exception to this rule is a problem that I have worked on with other students previously and left unfinished, since usually the first step a new collaborator takes on a problem is to change the direction of inquiry.

This is frequently a difficult requirement to fulfill, since I have more knowledge of my research areas than my students, and I guide the choice of area we'll be working in. Certainly the field of graph theory helps in this regard. A distinctive feature of the field is that there are hundreds of distinct subfields with different kinds of research problems that require different techniques of solution. It is relatively easy to find a new corner of a particular small subfield which is unexplored by anyone, and in particular by me. Graph theory is not unique in this respect, but perhaps in the minority among fields of mathematics. It also helps greatly to work

Received by the editor December 1, 2006.

in an area in which the problems are easily accessible and don't take a lot of startup time.

The student plays a central role in the selection of the problem. I never have a specific problem picked out before our research begins. Before the collaboration, I select one or several possible narrow fields of study. Ideally the field will have at least three papers published in the area, but usually less than twenty. With a few papers to read, we can get a good idea of what's been done in the field, and have a starting point to begin investigating problems in the area. On the other hand, with more than 20 papers already published, the problems start to become more difficult to find, and perhaps more importantly, the time it takes to read and absorb the required background material is prohibitively large (often for both me and my students).

I obtain these papers and skim through them, to determine what type of techniques are used in the proofs, and the range of possible unanswered or unasked questions. The project works much more smoothly when the techniques used in past research are accessible to the student, given his or her mathematical experience thus far. Often the topic of the project changes significantly enough in the first few weeks of work that the resources I have found become irrelevant, and we must return to perform a literature search again. This time the student takes an active role in the search, and gets to learn about mathscinet and requesting articles through interlibrary loan.

I believe that an undergraduate student's biggest strength as a researcher is that she or he is able to ask creative questions in a way that no other researcher in the field has done. Since they are new to the particular research area, undergraduate students are not familiar with the standard form of research problems in a given field, and often their questions are ones I would never have thought to ask myself. Indeed, since all of the problems I've worked on with undergraduate students have been new problems, no other mathematician has ever thought of them either. I'll give a few examples of this process of finding a problem from my past work with students.

Critical Pebbling Numbers of Graphs. Given a set of pebbles distributed on the vertices of a graph, a *pebbling step* takes two pebbles from one vertex and replaces one at an adjacent vertex. A distribution D of pebbles is *solvable* if, starting from D, a pebble can be moved to any specified vertex by a sequence of pebbling steps. The *pebbling number* $p(G)$ of a connected graph is the smallest number of pebbles such that every distribution with $p(G)$ pebbles is solvable. The (*global*) *critical pebbling number* $c(G)$ is the largest size of a minimally solvable distribution [**1**].

This graph parameter was defined by Courtney Gibbons, Erick Paul and myself in a research project in the summer of 2004. I started by suggesting a few possible research areas, including graph pebbling. I had seen a few talks on graph pebbling, found them interesting, and given a talk about pebbling in my department colloquium series. I used a survey paper on graph pebbling to write my talk, and I gave this paper to Courtney and Erick to study [**2**]. After reading through the paper, they decided they were interested in looking more closely at greedy pebbling and greedy graphs. We worked through some examples, and discovered that some of our initial assumptions about greedy pebbling were false, corresponding to a modified version of the definition of pebbling. We reworked this new definition for a few weeks, until we obtained the definitions of critical pebbling numbers in our paper.

Obstacle Numbers of Graphs. Given a set of vertices in the plane, and a set of (possibly non-convex) disjoint polygons in the plane called *obstacles*, we put a straight-line edge between two vertices if it does not intersect any obstacles. Given a graph G, the smallest number of obstacles required for such a representation of G is called the *obstacle number* of G.

This graph parameter was defined by Arden Brookstein, Richard Marcus, Andrew Ravits and myself during a mathematics research course I taught in the spring of 2006. I had eight students in this course, and at the beginning of the course I selected seven possible topics for research. I pitched each of the topics to the class on the first day, and they formed groups around the topics they were interested in. This particular research group selected the field of visibility graphs (somewhat broad considering my constraints), and started by looking at five papers on bar visibility graphs and rectangle visibility graphs. During a discussion about one of these papers, Richard remarked that lines of sight in a visibility representation reminded him of lines of sight in a video game, in which obstacles block visibility between opposing players. This led the team to the definition of obstacle numbers of graphs. The problem of finding a graph with large obstacle number carried the group through the course, and I have started investigating related problems with research colleagues since the course ended.

There are a couple big advantages to these requirements for the selection of a problem. First, as I discuss below, having been placed on roughly equal footing, I am free to involve myself as a full member of the research team. I don't feel the need to withhold information from my students to insure that they get the complete research experience. This means that we can accomplish more than we would otherwise. Second, with a problem which has never been studied, any progress is new, and it is much easier to find new results if the questions have not already been asked by other researchers.

2. The Research

Goals for a Successful Project. There are many measurements of the success of a student research project. I believe that if the student learns some mathematics, changes her or his attitude towards doing research in mathematics, or just enjoys working on a mathematics problem, then the project is successful. By this measure, all of my projects have been successful. Unfortunately, oftentimes a more objective measure of success is needed. Many attendees at the Conference on Promoting Undergraduate Research in Mathematics spoke of an increase in percentages of students enrolling in graduate school in mathematics as being a useful measurement of the success of a Research Experience for Undergraduates program. My ideal goal for a research project with an undergraduate student is that my student and I obtain original results and write an article for publication in a journal. This is a somewhat selfish goal on my part, since by this measure, a successful project means a publication for me, but it is one that every student I have talked to so far has been happy to embrace as his or her own as well. Indeed, many students are much more motivated to work hard with this goal in mind. Using this measure, about half of my student research projects have been successful so far.

Commitment to time. One clear pattern has emerged in my research with students. No other factor (level of student experience, difficulty of the research problem, area of research) matters more in predicting the success of the project than the amount of time the student commits to the project. Due to unpredictable variations in the difficulty of the problem, my projects have taken widely varying amounts of time, but a crude estimate is that all my student projects which produce a submitted article have resulted from at least 100 hours of work on the part of both my student and myself, and the other projects have generally taken less time. A student I have written two articles with estimates that we worked on each of our projects 20 hours a week for most of the summer, followed by additional time during the school year. Again, while this may seem like a big time commitment, it is time that I see myself investing not only in the quality of my students' education, but also in my own research program.

Since I have determined this rough guideline, I tell all my research students about it at the beginning of our project. Each student I talk to makes a decision about their commitment to time at the beginning of our collaboration, and most of those who decide they are willing to commit the time to achieve original results and a paper accomplish that goal.

Working as a team. My goal in research with students is that we work together as a research team, with each team member having equal control over the direction of the team. Most or all of the work is done together, when all team members are thinking about the problem in the same room. While I often assign tasks for students to complete between meetings, sometimes quite ambitious tasks, most of the time students use these tasks as a way to clarify and articulate their ideas and frustrations to be considered at the next meeting. The time I spend outside meetings thinking about the problem varies depending on how much progress we're making.

In all of my collaborations, both with students and colleagues, it's difficult or impossible to say how much each member of the collaboration contributed to the results. In a good collaboration, the end product would have been impossible without the work of any one of the team members. This has been true of most of my student research collaborations. While different members of the team had different strengths, the results would not have been possible without the participation of all collaborators.

Follow through. Any type of research with undergraduate students has the additional difficulty that the student will not be a student for a very long time. For example, if the research takes place over the summer before a student's senior year, and is submitted for publication in September of their senior year, it is very likely that the paper will not make it through the refereeing process until after the student graduates and probably moves out of town. After the student moves, regular meetings are impossible, and the remaining communication must be done through e-mail. Just as with any undergraduate research, if the team wants to submit an article, the professor must take on more responsibilities, including more of the revising of the article based on a referee's report. When I start a project, I commit to that extra work possibly some years down the road, but with the extra commitment comes the extra reward of an additional publication.

3. Results

I am very happy with the outcomes of my research collaborations with students. It is true that research with students takes a large amount of my time, and sometimes it doesn't work out. A student might feel pressures from other obligations, and run out of time to get anywhere on a research project. On the other hand, the model that I have been following for the past six years for research collaborations with students has many advantages. First, about half of my research collaborations with students have produced results significant enough to submit for publication, and these submitted papers constitute about half of my submitted research articles so far. Second, while some students and faculty might feel that undergraduate research is only an approximation of real mathematical research, by its nature there is essentially no difference between the research students do under my model and the research done by professional mathematicians. Lastly, and perhaps most importantly, the research I have done with students has been a lot of fun! It is difficult to find a collaborator who is more excited just to be involved in a research experience than the typical junior mathematics major thinking about a career in mathematics, and that excitement is infectious. I believe that I've found a model that works for me, and I plan to continue doing it for as long as I get the chance.

References

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