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Reflections on Undergraduate Research

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1. How it started

In 1987 I was hired as professor of Mathematics at California State University Fullerton, a predominantly undergraduate institution. The Administrators, from the President of the University to the Dean of the School of Art and Science and the Chair of the Mathematics Department, were placing a very strong emphasis in a combination of research and excellence in teaching. Financial support was available in the form of release time and paid participation to national and international meetings. I realized that the following two activities could provide a winning combination to achieve the goals supported by the university.

1. Publish simple and easily readable papers on topics of interests to our students.
2. Do research with our best undergraduates on topics that were accessible to them and of interest to me.

I realized that these two activities were going to absorb my energy and to affect negatively my traditional professional research, but I decided that they were worthy to pursue and I embarked in them fully aware of the potential risks and rewards. By the time I left California State University Fullerton in the year 2000 I had published collaborative work with six students: David Marshall [**16**], Mai Dang [**3, 4**], Tania Seph [**4**], Bethany Johnston [**14**], Gary Michaelian and Suzanne Sindi [**17**]. David and Suzanne would go on to earn a Ph.D. in Mathematics. David's thesis advisor was Prof. William McCallum from the University of Arizona in Tucson, and Suzanne's advisor was Prof. Jim Yorke from the University of Maryland, College Park. Gary earned a Ph.D. in Physics from UC Irvine. The others are teaching mathematics in community colleges or high schools in California.

I had also published several papers to meet the first goal of my agenda. My co-authors were Gerald Gannon ([**9, 10, 11, 12, 13**]), William Gearhart [**8**] and Harris Schultz [**18**].

In the fall of 2000 I arrived at Claremont McKenna College (CMC), following a special invitation from the administration. Since CMC is an undergraduate institution, I continued with my research collaboration with students. I collaborated with Carrie Staples [**7**] and with three undergraduate students: Adam Cox, Christopher

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Jones, and Alison Westfahl [**1**]. All these students are or are going to be graduate students in Mathematics, or Economics, or Law.

Am I done collaborating with undergraduates? Not at all. In my fall 2006 classes in Multivariable Calculus and Differential Equations I mentioned open problems I would be happy to solve in collaboration with some of my students. One of them, from my Multivariable Calculus, is very interested in the problem I presented and want to work with me on its solution. The future is open and promising!

Let me share with you a bit of my life as a mathematician. I received my degree from the University of Firenze in 1966 and my thesis advisor was Prof. Roberto Conti, who passed away about a month ago. I wrote my thesis on an inequality proved by De La Vallée Poussin and the result I obtained was published [**15**]. I became Assistant Professor at the University of Firenze in 1967 and Full Professor in 1976. I came to the USA for family reasons since my wife is from California. I was first Professor at Bryn Mawr College from 1979 to 1987 and in the fall of that year I arrived at California State University Fullerton (CSUF). To make my transition easier the CSUF Administration granted me tenure upon entrance.

Many colleagues regarded my move from Bryn Mawr College to CSUF as a step down, but I did not perceived it that way. I knew that at Fullerton my influence in the life of the students taking my classes could be far greater than anything I could ever accomplished at Bryn Mawr. In Italy private schools are the natural venue for students with lower than average skills and rich parents. University professors regard with a degree of skepticism every student coming from a private school. Hence, for me, teaching in a public university, was like "going back home." I had always some degree of uneasiness about investing my energies and my knowledge with students coming from financially able families. I felt that Fullerton was the right place for me, and I still remember, with great emotion and sense of accomplishments, what David Marshall's father told me the day his son graduated. "You have turned David's life around. Thank you."

Many readers would be curious to know how I found the time to work with my students, since the teaching load at Fullerton is four courses per semester. First, I am happy to credit Prof. Jim Friel, who at that time was Chair of the Mathematics Department, for repeatedly granting me release time so that I was required to teach four courses only a couple of times and every remaining semester I taught three (or even two) courses. Having said that, I must recognize that I met and studied with my students mostly during the summer. They were not paid for working with me, and I was not paid for working with them, but we studied together, sometimes at the department, other times at my house. Now and then I provided lunch. We all loved to do mathematics and we greatly enjoyed each other company.

This arrangement is impossible at a private institution, since the students go back home during the summer, unless they are provided an income that justifies their presence at the college where they study during the regular academic period. I learned this unpleasant fact the very first summer at Claremont McKenna College. All students went back home and I was left with no one to work with. I had problems to propose, but no one to listen. In Fullerton, it was different. The students lived at home, they usually took a summer job, but they set aside time to do research with me and to talk about mathematics. I really appreciated their willingness to come to my house in Claremont, so that I could avoid the trip of going to Fullerton, 22 miles away from my residence.

2. Topics

How did I select the topics of research I proposed? I did not have only one strategy. First, I did read a lot of mathematical journals, including, but not limited to, the American Mathematical Monthly, the College Mathematics Journal, Mathematics Magazine, Applied Mathematics Letters, etc. I paid particular attention to those papers that had open questions at the end and I asked myself if we could solve some of the questions left unanswered by the writer(s). I asked friends and colleagues to tell me of problems they found and they considered suitable for our students. When I was teaching, I always asked myself if some theorems could be generalized, or proved differently, or established for a different class of functions. I am happy to confess that I was never short of problems. In fact, I had abundance of them, many more than I could assign.

Before presenting a topic of research I did some preliminary investigation to check if the proposed question could be solved. I had the feeling that my students would be frustrated if their work would have come to an impasse. However, I was never sure, when we started, if we were going to be succesful. In some cases, in fact, the problem proved to be more difficult than I had anticipated, and we could not write a paper with our results. I am still puzzled by some of the problems and I do not have an answer, although I believe that the result we wanted to prove is indeed true. Let me give you an example.

In the book by R. Boas [**2**] I had found the celebrated universal chord theorem.

THEOREM 2.1. *Let $f : [0, 1] \to [0, 1]$ be continuous and such that $f(0) = f(1)$. Then for every positive integer n there are two points $0 \le x_1 < x_2 \le 1$ such that $f(x_1) = f(x_2)$ and $x_2 - x_1 = \frac{1}{n}$.*

The segment joining $(x_1, f(x_1)$ with $(x_2, f(x_2)$ is appropriately called a *horizontal chord* of f. An interesting complement to 2.1 is the following result

THEOREM 2.2. *Let $f : [0, 1] \to [0, 1]$ be continuous and such that $f(0) = f(1)$. Then for every $a \in (0, 1)$ the function has either one horizontal chord of length a or two different horizontal chords of length $1 - a$.*

Famous mathematicians have worked on these problems and published their discoveries either about proofs different from the existing ones or about different families of functions [**5, 6, 19, 20**].

I had the strong feeling that Theorem 2.2 could be established in higher dimension if the statement was modified in a suitable manner. In fact, the simple example of the function $f(t) = (\cos 2\pi t, \sin 2\pi t),\ t \in [0, 1]$ shows that some adjustments are needed. For example, *horizontal* may now be interpreted as *parallel* to the plane $z = 0$. I proposed the problem to Pilar Mata, a very talented student who did not want to pursue her Ph.D. in mathematics despite my encouragement to do so. Pilar and I worked on it for several weeks, and we did establish the validity of a higher dimensional version of Theorem 2.2 in many cases. However, a full proof of the result eluded us, and, in the end, we had to give up since we could not surmount the difficulties presented by some cases.

There was always some preliminary work to do with my students to bring them up to speed. For example, when Suzanne Sindi, Gary Michaelian and I were working on establishing necessary and sufficient conditions for pitchfork, transcritical, fold, and period doubling bifurcation [**17**], we had to review together the appropriate analytic conditions required by simple and double points of planar curves, including the exceptional case in which the tangent lines are vertical. I had to explain to them that given the set $S = \{(x, y) : f(x, y) = 0\}$ where f is C^n with n sufficiently large, and given a point $P \in S$ we can define the multiplicity of P as a solution of the equation $f(x, y) = 0$ by using the partial derivatives of f. In particular, P will be a double point if

$$\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0,$$

but not all second partials are 0 at P. We had also to understand why the slopes m of the two tangent lines to S at P are found by solving the quadratic equation

$$am^2 + 2bm + c = 0$$

where

$$a = \frac{\partial^2 f}{\partial x^2}(P),\ b = \frac{\partial^2 f}{\partial x \partial y}(P),\ c = \frac{\partial^2 f}{\partial y^2}(P).$$

Suzanne, Gary and I invested many hours in this problem. I purposely avoided to let the students know what I believed we were going to find. They discovered many, but not all, results on their own.

They also wanted to explore the situation in a higher dimensional setting, but, unfortunately, there was not time for this study. Both students left Fullerton to pursue their graduate studies, Gary at UCI and Suzanne at the University of Maryland, College Park.

Here is perhaps the most appropriate moment to underline that the collaboration with undergraduates is necessarily constrained by their four years training. After they go on to graduate studies, or to other activities, it is extremely difficult to establish a meaningful collaboration with them. The distance, the different interests, the pressure of the graduate program are all conspiring against any plan to continue a research program centered on problems more suitable for an undergraduate than for a graduate student. Therefore, the wisest move, when possible, is to recruit juniors and possibly even sophomores. In this case the instructor can be sure that the collaborative work will have the necessary continuity.

3. Recruiting

Hence, we naturally come to the question of how to find students who are willing and capable of collaborating with you. The "willing and capable" is an important combination. I have selected students from my own classes, I have asked the advice of other instructors, I have looked at the high school record, and at the record of courses taken by the student after their high school graduation. I realized, however, that nothing can replace direct talks with the undergraduates.

At least in one case, I made the choice simply because I had the strong feeling that the individual was very capable and the previous experiences were not representative of the student's real capabilities. Hence, for example, I invited David Marshall to work with me and I found out very soon that I had made the correct decision, even though his previous grades were average at best. I was probably influenced by my experience back in Italy. One of my classmates and later my collaborator was Massimo Furi, who did not shine in high school (Istituto Tecnico Industriale) simply because the mathematics he had to learn was uninspiring and deprived of ideas. At the university Massimo flourished and revealed an amazing potential.

Did I ever selected the "wrong" students? Yes, in two cases. One based on my own experience, and the other based on the advice of a colleague. It can happen. The undergraduates did not measured-up to my expectations and I worked for about a year with no tangible accomplishments.

In two cases I selected problems that were too difficult. We were not able to solve them and the results we obtained were too partial. It was a great disappointment for me, and an even greater disappointment for the students. Unfortunately, there is no way to find out, at the outset, if the result you want to establish is true or not. Moreover, even if we assume that the result is true, it could be very difficult to prove. I found out that missteps can be minimized but not completely avoided.

My name has always appeared among the list of authors of the paper written in collaboration with my undergraduates. Several reasons have dictated this strategy. I will mention only two. First, I felt that the inclusion of my name was more representative of the common efforts that went into the collaborative work. Second, I felt that adding my name would give more visibility to the papers and consequently, would be more beneficial to the students.

I encouraged the student's participation in regional and national meeting. I obtained from the administrations at Fullerton and at CMC the necessary funds to pay for transportation, registration, and lodging. I started an Undergraduate Student Poster Session at the spring meeting of the Southern California-Nevada Section of the MAA. This is now a well established tradition and the activity usually attracts thirty teams of undergraduates. The section pays for their registration and lunch.

For several years I organized a national Undergraduate Student Poster Session in conjunction with the annual joint meetings of the AMS and MAA in January. From humble origins the session had recently become one of the most visible and well attended activities of the joint meetings. I recently relinquished to Dr. Diana Thomas, of Montclair State University, the task of organizing the poster session and I understand that the Executive Committee of the MAA would like to open the participation to 250 teams of undergraduates. The amount of work required to reach this goal with a flawless organization is definitely challenging, but not impossible.

Let me close these remarks by repeating that my collaboration with undergraduates has been very rewarding and I would do it again. I feel that it should not be the primary research activity of a young faculty, since it requires a degree of experience and an investment of time that may prove to be prohibitive for a person at the beginning of the academic career. However, working with these young kids is fun and interesting. The influence a teacher can have in their future will probably last a lifetime.

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