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Undergraduate Research at Canisius Geometry and Physics on Graphs Summer 2006

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1. Overview

This is the second year we have hosted the R.E.U. program at Canisius College. Two faculty members of Canisius College, Terrence Bisson and myself, were responsible for the program. It lasted for eight weeks and we had eight students participating.

The main focus of the program was research on graphs using methods from category theory, algebraic topology and physics. The students were divided in two groups of four students. The first group, directed by Terrence Bisson, worked on spectral properties of directed graphs and their covers. In particular, the group worked on the following problems:
- to what extent does the spectrum of the Cayley graph of a finite abelian group determine the graph up to graph isomorphism? Abelian groups were chosen because the eigenvalues of their Cayley graphs can be expressed as sums of roots of unity. The problem was completely solved when the generating set has a small cardinality.
- In the previous summer, divisibility properties of Chebychev polynomials (of first and second type) where derived by looking at covers of directed path and circle graphs. Variations of this method produced more divisibility properties for Chebyschev - like polynomials. This summer one student derived closed formulas for the coefficients of these characteristic polynomials by reduction to a collection of solvable combinatorial problems.
- What is the effect on the spectrum of directed graphs under graph operations? Different graph operations (endo-functors on the category of directed graphs) were described (addition, multiplication, exponentiation, doubling) and the effect on the spectrum of the result was studied.
- What type of information can be derived from the spectra of the family of coset graphs of a finite group, and the coverings between them? The case for the symmetric group on 4 letters was studied in detail. The coset graphs of a group G were also applied to Burnside-Polya methods for computing the number of different colorings of figures with a G action.

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The second group, directed by me, worked on understanding the connections between the spectrum of the Laplacian of a graph and its combinatorial properties. More specifically the problems considered were the following:

- Give the definition and the properties of a Kazhdan graph. Kazhdan groups are defined as groups whose trivial representation is isolated in the space of representations. That puts certain restrictions on the spectrum of the Cayley graphs of the group. In particular, the spectrum of any finite cover of a graph covered by the Cayley graph of the group has a 'spectral gap' (the difference between its first and second eigenvalue) that only depends on the group and not on the cover. That property was used in the construction of infinite families of expanders. In this project the students defined a Kazhdan graph to be a graph that satisfies a similar spectral property, they constructed non-Cayley graphs that are Kazhdan graphs, and they prove the expansion properties of finite graphs covered by them.
- Generalize the definition of the zig-zag product of graphs. Zig-zag products were used to construct infinite families of expanders combinatorially, without using algebraic methods. The students gave a categorical description of the construction that generalizes the classical construction. Then, they computed the spectral properties of the construction. Also, they described the covering properties of the generalized zig-zag products and used it to construct infinite families of graphs, each covering the previous one, that have the same spectrum.
- Define and compute Ihara's zeta functions for certain infinite graphs.In this project, the students computed the Ihara zeta function for certain infinite covers of graphs that can be expressed as limits of finite covers. In particular, they generalized Bass' formula of the zeta function to infinite graphs, by replacing the determinant by the trace of an operator to certain operator algebra.
- Describe the combinatorial Laplacian of "homesick" random walks on graphs. In this project, the students studied the combinatorial Laplacian of the homesick random walk on the infinite path and they showed that it is equal to the Laplacian of the regular random walk on a regular tree, when the parameter is an integer. Using this result, they computed the spectrum of the Laplacian and the heat kernel of the homesick random walk on lattices.
- Describe vanishing results for repeated exotic Nil-groups in algebraic K-theory. One of the students recruited had a strong background in algebra and category theory. The project that he worked on was on certain computations in algebraic K-theory. More specifically, when the K-theory of push-out diagrams of rings is computed, there is an "error term" that determines how far is K-theory from being a homology theory. In this case, this groups are called Waldhausen's Nil-groups. The computations done this summer are on Waldhausen's Nil-groups when the base ring is also a push-out in the category of rings. Repeated Nil-groups are derived this way and certain vanishing results are proved.

The students maintained a wiki site where they recorded most of the calculations, experiments and partial results they derived. There is a plethora of data and speculations in the site: http://wiki.canisiusmath.net.

2. The Program

The recruitment of the students was done by distributing a flyer to the nearby colleges and universities. To reach more students, flyers were distributed in conferences attended by Canisius faculty, like the Nebraska Conference for Women in

Mathematics. Four of the students were from the western part of the country, one from the south and three from the wider area around Buffalo. Four students were female and four male. The stipend was $2000. Accommodations were included in a dormitory at Canisius College (Eastwood Hall). Also, meals were included and they were provided by the food services of the college. The College provided for us classrooms for our meetings and computer access for all our needs. The weekends were spent in social gatherings, organized by the Canisius faculty or the organizers, or by visiting local attractions like Niagara Falls.

Even though this is the second year we are hosting the program, we manage to recruit very strong students. Three out of the six seniors are applying to graduate programs in mathematics and they have very good chances to be accepted and succeed. The other three seniors will follow a career as math educators. The two who were not seniors have the interest and the ability to become researchers in mathematics.

The students would meet with us every morning and work together until late afternoon, with a lunch break. Usually, they discussed the work they did last evening and they asked question on how to continue their research. The two groups worked separately but they interacted and exchanged ideas all the time. There were invited speakers that gave talks to the students on different subjects. The speakers were from Canisius, SUNY Buffalo and the University of Rochester.

3. Conclusion

Overall, we are very satisfied with our recruitment process and the quality of the work produced this summer. Two papers have been submitted for publication. One of them and a third paper are posted in the archive preprint series. The students presented their result in the Young Mathematicians Conference at Ohio State University in early August. They received very positive comments on their presentations.

Judging from the participants' comments, the organizers believe that the program had a strong impact in their decision to pursue a career as research mathematicians or educators.

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