Math 5372 Exam 1 Review
Spring 2009 Solutions

1. **How many abelian groups are there of order $2^5\, 3^2\, 5^3$?**

There are two abelian groups of order p^2, and three of order p^3. With three primes, this gives 2×2×3 = 12 possible abelian groups.

2. **Let G be a group and let H be a normal subgroup of G. Prove or give a □counter example to the following: If H is abelian and G/H is abelian, then G is abelian.**

This is false. An example would be $G = D_4$ (nonabelian) with normal subgroup $H = C_4 = \langle R_{90} \rangle$ (abelian). Here $D_4/C_4 \cong Z_2$, an abelian group.

3. **Let** $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a \neq 0,\ a, b \text{ are real} \right\}$. **G is a group. Let** $\alpha: G \to R^*$ **be defined by** $\alpha\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a$. **Let** $H = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \;\middle|\; c \text{ is real} \right\}$.

a. **Show that H is a normal subgroup of G.**

If $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ then $g^{-1} = \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix}$. With $h = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, we have $g^{-1}hg$

$= \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b + c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c/a \\ 0 & 1 \end{pmatrix} \in H$, so H is normal.

b. **What isomorphism of groups results from the First Isomorphism theorem using G, R^*, and α?**

First, α is a homomorphism: $\alpha\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\right) = \alpha\begin{pmatrix} ac & ad + b \\ 0 & 1 \end{pmatrix} = ac$

$= \alpha\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \alpha\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$. Next, α is onto: if $r \neq 0$, then $\alpha\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} = r$.

kernel of α: We want $\alpha\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ = identity in R^*, or 1, so we need $a = 1$.

That is, $\ker(\alpha) = H$. (This also shows that H is normal, by the way−−H is the kernel of a homomorphism). Finally, the First Isomorphism says $G/\ker(\alpha) \cong \alpha(G)$, so in this case, $G/H \cong R^*$.

4. Let $G = Z_{40} \oplus Z_{50} \oplus Z_{90} \oplus Z_{100}$.
   a. **Express G as a direct product of cyclic groups of prime power order.**

$$G = Z_{40} \oplus Z_{50} \oplus Z_{90} \oplus Z_{100} \cong Z_8 \oplus Z_5 \oplus Z_2 \oplus Z_{25} \oplus Z_2 \oplus Z_9 \oplus Z_5 \oplus Z_4 \oplus Z_{25}$$
$$\cong Z_2 \oplus Z_2 \oplus Z_4 \oplus Z_8 \oplus Z_9 \oplus Z_5 \oplus Z_5 \oplus Z_{25} \oplus Z_{25}$$

   b. **Express G as a direct product in the form $Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_k}$ where m_1 is a multiple of m_2 is a multiple of $\cdots$ is a multiple of m_k.**

$$G \cong (Z_8 \oplus Z_9 \oplus Z_{25}) \oplus (Z_4 \oplus Z_{25}) \oplus (Z_2 \oplus Z_5) \oplus (Z_2 \oplus Z_5)$$
$$\cong Z_{1800} \oplus Z_{100} \oplus Z_{10} \oplus Z_{10}$$

5. **How many homomorphisms are there from Z_{40} onto Z_{40}? Z_{40} into Z_{40}?**

This was kind of a silly question. Homomorphisms from Z_{40} onto Z_{40} are isomorphisms, and we know that there are exactly $\varphi(40) = 16$ of them (they are all of the form $\alpha(x) = rx$, for $r \in U(40)$).

For into homomorphisms, it turns out that for any a in Z_{40}, $\beta(x) = ax$ is a homomorphism, and all homomorphisms have this form, so there are exactly 40 homomorphisms.

A more interesting problem might have been to talk about homomorphisms from Z_{20} to Z_{15}, say. In this case, given $\varphi: Z_{20} \to Z_{15}$, if $\varphi(1) = a$, then $\varphi(x) = ax$ for all x. When is such a formula a homomorphism? We know that $|\varphi(1)|$ has to be a divisor of $|1| = 20$, and also that $|a|$ has to be a divisor of 15. This means $|a|$ must be a divisor of 5. If $|a| = 1$, then $a = 0$, and this works. If $|a| = 5$, then $a = 3, 6, 9$, or 12. It turns out that all these work, so there are a total of 5 homomorphisms from Z_{20} into Z_{15}, and none of them are onto.

6. **$Z \oplus Z$ is both a ring and a group.**
   a. **Which subgroups of the form $\langle(a, b)\rangle$ are subrings of $Z \oplus Z$?**

Obviously $\langle(a, b)\rangle$ is closed under subtraction so we need closure under multiplication. Elements in $\langle(a, b)\rangle$ have the form $x = (na, nb)$ for some n. We have $xy = (na, nb)(ma, mb) = (mna^2, mnb^2)$, and we need this to have the form (ka, kb), for some integer k. If $a = 0$ or $b = 0$, this will always be the case. What happens if neither a nor b is 0? $ka = mna^2$ says $k = mna$, so

$kb = mnab = mnb^2$ holds for all m, n only when a = b. Thus, the subrings have one of the following forms: <(a, 0)>, <(0, b)>, or <(a, a)>.

b. **What are the units of $Z \oplus Z$?**

We need (a, b)(c, d) = unity = (1, 1). This means we need ac = 1, bd = 1, so the units are (1, 1), (1, −1), (−1, 1), (−1, −1).

c. **What are the zero divisors of $Z \oplus Z$?**

(a, b)(c, d) = 0 = (0, 0) means we need ac = 0 and bd = 0. That is, a = 0 or c = 0 and a similar condition holds for b, d. The zero divisors: things of the form (a, 0) or of the form (0, b).

7. **Let R be a commutative ring with unity.**
   a. **If a is a zero divisor in R and b is any element of R such that $ab \neq 0$, prove that ab is a zero divisor.**

There must be some $x \neq 0$ with ax = 0. If $ab \neq 0$, then abx = (ax)b = 0, so ab is also a zero divisor. (The reason for the condition $ab \neq 0$ is that 0 is not considered to be a zero divisor.)

b. **Prove that if a is invertible in R, then a is not a zero divisor. Hint: suppose ax = 0 for some x in R.**

Let a be invertible and suppose ax = 0. Let b be the inverse of a so ab = 1. We have axb = 0b = 0, and axb = abx = 1x = x. That is, x = 0. Thus the only product with a that gives 0 is a×0, meaning a is not a zero divisor.

c. **Let R[i] = {a + bi | a, b are in R}. Prove that a + bi is a zero divisor in R[i] only if $a^2 + b^2 = 0$ or $a^2 + b^2$ is a zero divisor in R. Hint: $a^2 + b^2 = (a + bi)(a - bi)$.**

Suppose that a + bi is a zero divisor. by 7a, either (a + bi)(a − bi) is zero or it is also a zero divisor. If it is 0, then $a^2 + b^2 = 0$. Otherwise, $a^2 + b^2$ is a zero divisor in R.

8. **One of Z_5 [i] and Z_7 [i] is a field and the other is not. Determine which is which. Hint: use 7(c).**

A finite integral domain is a field, so we need only determine whether there are zero divisors or not. By problem 7c, this is the same as asking if $a^2 + b^2$ is either zero or a zero divisor in Z_5 or Z_7. Neither of these rings has zero divisors, so we only need to determine if $a^2 + b^2$ can ever be zero (without a, b both being 0). We have: $1^2 + 2^2 = 0$ in Z_5, so Z_5 [i] has zero divisors. (In fact, $(1 + 2i)(1 - 2i) = (1 + 2i)(1 + 3i) = 0$ in Z_5 [i], so Z_5 [i] is not a field.)

In Z_7, the squares are 0, 1, 2 ($= 3^2$) and 4, and the only sum of two of these that can be 0 is $0 + 0$, so the only way $a^2 + b^2 = 0$ is if both $a = 0$ and $b = 0$. This means Z_7 [i] has no zero divisors making it a field.