Math 5371 Final Exam Review
Spring 2009

1. An idempotent in a ring R is an element e such that $e^2 = e$. What are the idempotents in $Z \oplus Z$? $Z \oplus Z \oplus Z$?

2. Find the sizes of the following rings.
   a. $Z\left[\sqrt{-2}\right]/\langle \sqrt{-2} \rangle$ b. $Z\left[\sqrt{-2}\right]/\langle 2 + \sqrt{-2} \rangle$
   c. $Z\left[\sqrt{-2}\right]/\langle 1 + 2\sqrt{-2} \rangle$ (Hint: This is trickier than (b), but note that $-\sqrt{-2}\left(1 + 2\sqrt{-2}\right) = 4 - \sqrt{-2}$.)

3. Let R be the real numbers.
   a. Show that the function $\varphi: R \to C$, the complex numbers defined by $\varphi(p(x)) = p(i)$ (where $i^2 = -1$) is a ring homomorphism.
   b. Use (a) to show that $R[x]/\langle x^2 + 1 \rangle \cong C$.

4. Factor $x^9 - 1$ into irreducible polynomials over Q. Justify that each term is, in fact, irreducible.

5. $x^2 + 2x + 4$ fails to be an Eisenstein polynomial because $2^2 \mid 4$.
   a. Show that $x^2 + 2x + 4$ is irreducible over Q anyway.
   b. Give an example of a quadratic $x^2 + ax + b$ over Z which is reducible over Q, for which $a \neq 0$, $b \neq o$ and for some prime p, $p \mid a$ and $p^2 \mid b$.

6. Show that for any $n \geq 1$, there is an irreducible polynomial over Q of degree n.

7. $Z\left[\sqrt[3]{2}\right] = \left\{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in Z\right\}$.
   a. Show that $\langle \sqrt[3]{2} \rangle$ is a maximal ideal of $Z[\sqrt[3]{2}]$.
   b. Show that $\langle \sqrt[3]{4} \rangle$ is not a maximal ideal of $Z\left[\sqrt[3]{2}\right]$. What happens in $Q\left(\sqrt[2]{2}\right)$?

8. Show that $Z\left[\sqrt[3]{2}\right]$ is a Euclidean ring. Hint: use $N\left(a + b\sqrt[3]{2} + c\sqrt[3]{4}\right) = \left|a^3 + 2b^3 + 4c^3 - 6abc\right|$. You may use the properties: N(x) is always an ordinary integer, N(x) = 0 only if x = 0, and N(xy) = N(x)N(y).

9. Show that $Q(\sqrt[6]{2}) = Q(\sqrt{2}, \sqrt[3]{2})$.

10. Find the splitting field for $x^4 - x^2 + 1$ over each of the following fields. (In each case, you may write the answer in the form $F(a, b, c, \cdots)$ where $a, b, c, \cdots$ are zeros of specified irreducible polynomials.)
a. Q b. R c. Z_2 d. Z_3 e. Z_5

11. There is still room for elementary algebra in a class like this. Suppose that p(x) is an irreducible cubic over Q. If p(x) has one real root, show that the splitting field, E, of p(x) over Q has [E, Q] = 6.

12. Let $a = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$ and $b = \sqrt{2 + \sqrt{2}}$.
Find, with proof, the minimal polynomials of a, b, a + b over Q.

13. Find, with proof, the minimal polynomial of $\sqrt{2} + \sqrt[3]{2}$ over each of the following fields:
a. $Q(\sqrt[3]{2})$ b. $Q(\sqrt{2})$ c. Q

14. Let F be any field of characteristic $\neq 2$ and let E be an extension of F such that $[E: F] = 2$. Show that $E = F(\sqrt{a})$ for some $a \in F$. Hint: complete the square in an appropriate quadratic.