1. Find the coefficient of $x^{15}$ in each of the following:

(a) $\frac{1}{(1-x)^6}$

**Solution:**

$$\frac{1}{(1-x)^6} = \sum_{k=0}^{\infty} \binom{k+5}{k} x^k$$

so the coefficient of $x^{15}$ is $\binom{20}{15} = \binom{20}{5} = 15,504$.

(b) $\frac{(1+x^2)^3}{(1-x)^6}$

**Solution:**

$$\frac{(1+x^2)^3}{(1-x)^6} = (1 + 3x^2 + 3x^4 + x^6) \sum_{k=0}^{\infty} \binom{k+5}{k} x^k.$$  

The coefficient of $x^{15}$ will be $\binom{20}{5} + 3 \binom{18}{5} + 3 \binom{16}{5} + \binom{14}{5} = 56,314$.

(c) $15!(1+x^2)^3 e^{6x}$

**Solution:** The coefficient of $x^n$ in $e^{6x}$ is $\frac{6^n}{n!}$. We have $15!(\frac{6^{15}}{15!} + 3\frac{6^{13}}{15!} + 3\frac{6^{11}}{15!} + \frac{6^9}{9!})$, or $6^{15} + 3 \cdot 6^{13}P(15, 2) + 3 \cdot 6^{11}P(15, 4) + 6^9P(15, 6)$.

2. Find a closed form expression for the generating function for the sequence:

(a) 5, 5, 5, 5, \cdots

**Solution:** Answer: $5 + 5x + 5x^2 + \cdots = \frac{5}{1-x}$.

(b) 5, 0, 5, 0, 5, 0, \cdots

**Solution:** Here, the generating function is $5 + 5x^2 + 5x^4 + \cdots$, which is the same as in part (a), but with $x$ replaced by $x^2$, so the answer is $\frac{5}{1-x^2}$.

(c) 1, 6, 17, 34, \cdots (the $k$’th term is $1 + 2k + 3k^2$)

**Solution:** We could break up the problem to finding 3 different generating functions: $\sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty} 2kx^k + \sum_{k=0}^{\infty} 3k^2x^k$. But it is usually easier to use the trick
that \( \sum_{k=0}^{\infty} \binom{m+k}{k} x^k = \frac{1}{(1-x)^{m+1}} \). To use the trick, try to write \( 1 + 2n + 3n^2 \) as a sum of binomial coefficients of the appropriate type. This is another version of an undetermined coefficients problem: Write

\[
1 + 2n + 3n^2 = A + B \binom{n+1}{1} + C \binom{n+2}{2}.
\]

When \( n = 0 \) we get \( A + B + C = 1 \), when \( n = 1 \), \( A + 2B + 3C = 6 \), and when \( n = 2 \), \( A + 3B + 6C = 17 \). This system has solution \( A = 2, B = -7, C = 6 \) so

\[
\sum_{k=0}^{\infty} (1 + 2k + 3k^2)x^k = \sum_{k=0}^{\infty} \left( 2 - 7 \binom{k+1}{1} + 6 \binom{k+2}{2} \right) x^k
= \frac{2}{1-x} - \frac{7}{(1-x)^2} + \frac{6}{(1-x)^3}.
\]

3. Use the answer to part (c) above to find a formula for \( 1 + 6 + \cdots + (1 + 2n + 3n^2) \).

**Solution:** We need the coefficient of \( x^n \) in \( \frac{1}{1-x} A(x) \), where \( A(x) \) is the generating function for the sequence in part (c). That is, we need the coefficient of \( x^n \) in

\[
\frac{2}{(1-x)^2} - \frac{7}{(1-x)^3} + \frac{6}{(1-x)^4},
\]

or

\[
2 \binom{n+1}{1} - 7 \binom{n+2}{2} + 6 \binom{n+3}{3} = \frac{2n^3 + 5n^2 + 5n + 2}{2}.
\]

4. Find the generating function for each of the following problems. Write your answer as simply as possible (that is, \( 1 + x^2 + x^3 + x^4 + \cdots \) should be written \( \frac{1}{1-x} - x \) or \( \frac{1-x+x^2}{1-x} \). Note: Just find the generating function (ordinary or exponential,) don’t solve for \( x^n \).

(a) In how many ways can \( n \) balls be put into 5 boxes if there is at least one ball in the first box and at most 3 in the second?

**Solution:****

\[
\frac{x}{1-x} (1 + x + x^2 + x^3) \frac{1}{1-x} \frac{1}{1-x} \frac{1}{1-x} = \frac{x(1 + x + x^2 + x^3)}{(1-x)^4}, \text{ or } \frac{x - x^5}{(1-x)^5}.
\]

Page 2
(b) How many sequences made up of $a, b, c, d, e$ of length $n$ are there if there is at least one $a$ and at most 3 $b$'s?

Solution:

$$ (e^x - 1) \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \right) e^x e^x e^x = \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \right) (e^{4x} - e^{3x})$$

(c) How many ways can $n$ balls be put into 5 boxes if the first box has an even number of balls and the second, an odd number?

Solution: The generating function for an even number of balls in a box is $1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$. For an odd number of balls, you just multiply this by $x$. The generating function we seek is

$$ \frac{x}{(1-x^2)^2(1-x)^3}.$$ 

(d) How many sequences made up of $a, b, c, d, e$ of length $n$ are there if there must be an even number of $a$’s and an odd number of $b$’s?

Solution: Here, the generating functions for even/odd are $\frac{e^x + e^{-x}}{2}$ and $\frac{e^x - e^{-x}}{2}$. The generating function we seek is

$$ \frac{e^x + e^{-x}}{2} \cdot \frac{e^x - e^{-x}}{2} e^x e^x = \frac{e^{5x} - e^x}{4}.$$ 

(e) How many ways can $n$ balls be put into 5 boxes if no box has exactly 2 balls?

Solution: For one box it would be $1 + x + x^3 + x^4 + \cdots = \frac{1}{1-x} - x^2 = \frac{1 - x^2 + x^3}{1-x}$. Probably the first formula is better. Our generating function is

$$ \left( \frac{1}{1-x} - x^2 \right)^5.$$ 

Page 3
(f) How many sequences made up of \(a, b, c, d, e\) of length \(n\) are there if you can’t have exactly 2 of any letter?

\[
\text{Solution:} \quad \text{Here there really only is one answer: } \left( e^x - \frac{x^2}{2} \right)^5.
\]

5. Find the solutions to the problems in question 1.

(a) In how many ways can \(n\) balls be put into 5 boxes if there is at least one ball in the first box and at most 3 in the second?

\[
\text{Solution:} \quad \text{We seek the coefficient of } x^n \text{ in } \frac{x - x^5}{(1 - x)^5}. \text{ This will be }
\]
\[
\binom{n - 1 + 4}{4} - \binom{n - 5 + 4}{4} = \binom{n + 3}{4} - \binom{n - 1}{4}.
\]
I will note that this formula is correct if \(n > 0\) (rather than just \(n > 5\)) because \(\binom{n}{k}\) is defined to be 0 \(k > n \geq 0\).

(b) How many sequences made up of \(a, b, c, d, e\) of length \(n\) are there if there is at least one \(a\) and at most 3 \(b\)’s?

\[
\text{Solution:} \quad \text{Here we are stuck with a rather complicated expression. We need } n! \text{ times the coefficient of } x^n \text{ in } 
\frac{x}{(1 - x)^2} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \right) \left( e^{4x} - e^{3x} \right). \text{ We get }
\]
\[
4^n - 3^n + n(4^{n-1} - 3^{n-1}) + \frac{n(n - 1)}{2} (4^{n-2} - 3^{n-2}) + \frac{n(n - 1)(n - 2)}{6} (4^{n-3} - 3^{n-3}).
\]
I don’t know of any good way to simplify this.

(c) How many ways can \(n\) balls be put into 5 boxes if the first box has an even number of balls and the second, an odd number?

\[
\text{Solution:} \quad \text{We need the coefficient of } x^n \text{ in } \frac{x}{(1 - x^2)^2(1 - x)^3}. \text{ This is just a hard problem. If we were to use partial fractions, we would have }
\]
\[
\frac{x}{(1 - x^2)^2(1 - x)^3} = \frac{x}{(1 + x)^2(1 - x)^5} = \frac{1}{64} \left( -\frac{3}{1 - x} - \frac{4}{(1 - x)^2} + \frac{4}{(1 - x)^3} + \frac{4}{(1 - x)^5} - \frac{3}{1 + x} - \frac{2}{(1 + x)^2} \right).
\]
Our coefficient would be
\[
\frac{1}{64} \left( -3 - 4(n + 1) + 4 \left( \frac{n + 2}{2} \right) + 4 \left( \frac{n + 4}{4} \right) - 3(-1)^n - 2(n + 1)(-1)^n \right).
\]

(d) How many sequences made up of \(a, b, c, d, e\) of length \(n\) are there if there must be an even number of \(a\)'s and an odd number of \(b\)'s?

**Solution:** This problem is much easier as we want \(n!\) times the coefficient of \(x^n\) in \(\frac{e^{5x} - e^x}{4}\), or \(\frac{5^n - 1}{4}\).

(e) How many ways can \(n\) balls be put into 5 boxes if no box has exactly 2 balls?

**Solution:** We have
\[
\left( \frac{1}{1 - x} - x^2 \right)^5 = \frac{1}{(1 - x)^5} - \frac{5x^2}{(1 - x)^4} + \frac{10x^4}{(1 - x)^3} - \frac{10x^6}{(1 - x)^2} + \frac{5x^8}{1 - x} - x^{10}
\]
so if \(n > 10\) we get
\[
\left( \frac{n + 4}{4} \right) - 5 \left( \frac{n + 1}{3} \right) + 10 \left( \frac{n - 2}{2} \right) - 10(n - 5) + 5,
\]
with various other answers for smaller \(n\).

(f) How many sequences made up of \(a, b, c, d, e\) of length \(n\) are there if you can’t have exactly 2 of any letter?

**Solution:** This time, the two cases are fairly similar in complexity. We have
\[
\left( e^x - \frac{x^2}{2} \right)^5 = e^{5x} - 5x^2e^{4x} + 10x^4e^{3x} - 10x^6e^{2x} + 5x^8e^x - x^{10}
\]
so for \(n > 10\) the answer is
\[
5^n - 5n(n - 1)4^{n-2} + 10P(n, 4)3^{n-4} - 10P(n, 6)2^{n-6} - 5P(n, 8).
\]
Technically, we only need worry about \(n = 10\) as an exceptional case here, since \(P(n, k) = 0\) if \(n < k\). When \(n = 10\), we must subtract 10! from the expression above (with \(n\) replaced by 10 everywhere).
6. Use generating functions to find the number of ways to put \( n \) balls into 3 boxes if the first two boxes can hold at most 10 balls. Your answer will vary depending on the size of \( n \). For large \( n \), simplify your answer as much as possible. Can you prove your answer (for large \( n \)) by an easy counting argument?

**Solution:** The generating function is 
\[
(1 + x + \cdots + x^{10})^2(1 + x + x^2 + \cdots),
\]
or 
\[
\frac{(1 - x^{11})^2}{(1 - x)^3} = \frac{1 - 2x^{11} + x^{22}}{(1 - x)^3}.
\]
The coefficient of \( x^n \) in this expression, for large \( n \) is 
\[
\binom{n + 2}{2} - 2\binom{n - 9}{2} + \binom{n - 21}{2}.
\]
Expanding this out and simplifying, you get a constant, 121. The reason: for large \( n \), there are 11 choices for how many balls to put in the first box and 11 for the second. What’s left goes in the third box. So if \( n \geq 22 \) we get \( 11 \times 11 = 121 \) possible combinations.

7. Find recurrences, with initial conditions, for each of the following problems.

(a) How many \( n \)-digit ternary sequences do not contain consecutive 0’s?

**Solution:** This is kind of easy after the last few questions. The reason: This problem (both parts) came from a past exam. If the first position is a 1 or a 2, the remaining \( n - 1 \) positions just have to avoid consecutive 0’s. This can happen in \( 2a_{n-1} \) ways. If the first position is a 0 then the second position must contain 1 or 2, and the remaining \( n - 2 \) positions just need to avoid consecutive 0’s. The relation is \( a_n = 2a_{n-1} + 2a_{n-2} \).

For initial conditions, the empty sequence does not contain consecutive 0’s, nor does any sequence of length 1 so we have \( a_0 = 1, a_1 = 3 \). If you worry about whether the empty sequence has some property, you can find \( a_1 \) and \( a_2 \) instead. In this case there is only one bad string of length 2 so \( a_2 = 9 - 1 = 8 \).

(b) How many \( n \)-digit ternary sequences contain consecutive 0’s?

**Solution:** Usually, the recurrence for this kind of problem is the same as for without, except that there will be an inhomogeneous term. The count goes like this: If the first position is a 1 or a 2, then we need to pick up consecutive 0’s among the remaining \( n - 1 \) positions for a count of \( 2a_{n-1} \). If the first position is a 0 but the second is 1 or 2, we still need consecutive 0’s, and they must come from the remaining \( n - 2 \) positions for a count of \( 2a_{n-2} \). But here, we have another case: If the first two positions are 0’s, we have our consecutive
0’s and the remaining $n - 2$ positions are arbitrary. The recurrence is $a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$. Here, the initial conditions are $a_0 = 0, a_1 = 0,$ or if you prefer, $a_1 = 0, a_2 = 1.$

8. Last time I taught this course, I put the following problem on Exam 2, but the class found it fairly hard (under exam conditions). Suppose we form a sequence out of the numbers 0, 1, 2, 3, 4, 5.

(a) Find a recurrence for $a_n$, the number of such sequences of length $n$ which don’t contain consecutive numbers not divisible by 3. That is, sequences like 10403 are ok but 13423 is not because 4 and 2 are consecutive numbers not divisible by 3.

Solution: 0 and 3 are divisible by 3, and 1, 2, 4, 5 are not. If the first position is 0 or 3, we just need to avoid consecutive numbers not divisible by 3 in the remaining $n - 1$ positions, contributing $2a_{n-1}$. If the first digit is 1, 2, 4, or 5, then the next digit must be 0 or 3, and we need to avoid numbers not divisible by 3 in the remaining $n - 2$ positions. This gives a count of $4 \cdot 2a_{n-2}$ for this case. The recurrence is $a_n = 2a_{n-1} + 8a_{n-2}$.

(b) Find the initial conditions for $a_n$.

Solution: As usual, sequence of length 0 or 1 are to short to cause problems so $a_0 = 1, a_1 = 6$. If you want $a_1$ and $a_2$ in stead, we must do a little more work. There are 36 sequences of length 2, and $4 \cdot 4 = 16$ of them are bad so $a_2 = 20$.

(c) Solve the recurrence. There should be fractions but no radicals.

Solution: This was the point of the problem: I was proud of myself for coming up with a fairly natural recurrence that had a nice solution. The characteristic polynomial is $x^2 - 2x - 8 = (x + 2)(x - 4)$. The general solution is $a_n = A(-2)^n + B4^n$. The initial conditions tell us $A + B = 1, -2A + 4B = 6$ so $B = \frac{4}{3}, A = -\frac{1}{3}$. Our solution is $a_n = \frac{4^{n+1} - (-2)^n}{3}$.

(d) Find a recurrence for the number of such sequences of length $n$ which DO contain consecutive numbers not divisible by 3. Now 10403 is not ok but 13423 is.

Solution: If we start 0 or 3, we need consecutive numbers not divisible by 3 in the remaining $n - 1$ positions. If we start with 1, 2, 4, 5 but follow with 0 or 3, then we need to fulfill the requirements in the remaining $n - 2$ positions. Finally, if the first and second positions contain any of 1, 2, 4, 5, then the remaining
9. Solve the following recurrence relations.

(a) \( a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3} \), \( a_0 = 1, \ a_1 = 4, \ a_2 = 14 \).

**Solution:** The characteristic polynomial is \( x^3 - 3x^2 - 4x + 12 = (x-2)(x+2)(x-3) \). The general solution is \( a_n = A2^n + B(-2)^n + C3^n \). The initial conditions give \( A + B + C = 1, \ 2A - 2B + 3C = 4, \ 4A + 4B + 9C = 14 \), which has solution \( C = 2, \ A = -1, \ B = 0 \). The solution is \( a_n = 2 \cdot 3^n - 2^n \).

(b) \( a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3} + 3 \cdot 4^{n-2} \), \( a_0 = 1, \ a_1 = -2, \ a_2 = 6 \).

**Solution:** We try for particular solution \( a_n = P4^n \), which gives

\[
P4^n = 3P4^{n-1} + 4P4^{n-2} - 12P4^{n-3} + 3 \cdot 4^{n-2}.
\]

Dividing by \( 4^{n-2} \), we have \( 16P = 12P + 4P - 3P + 3 \), or \( P = 1 \). The general solution is \( a_n = A2^n + B(-2)^n + C3^n + 4^n \), and with the initial conditions, \( A + B + C = 0, \ 2A - 2B + 3C = -6, \ 4A + 4B + 9C = -10 \), and we have \( A = 1, \ B = 1, \ C = -2 \) so \( a_n = 4^n - 2 \cdot 3^n + 2^n + (-2)^n \).

(c) \( a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} \), \( a_0 = 1, \ a_1 = 2, \ a_2 = 6 \).

**Solution:** The characteristic polynomial is \( x^3 - 7x^2 + 16x - 12 = (x-2)^2(x-3) \). The general solution is \( a_n = (A + Bn)2^n + C3^n \), and the initial conditions tell us \( A + C = 1, \ 2A + 2B + 3C = 2, \ 4A + 8B + 9C = 6 \). The solution to these is \( A = -1, \ B = -1, \ C = 2 \) so \( a_n = 2 \cdot 3^n - (n+1)2^n \).

(d) \( a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} - 2^n \), \( a_0 = 4, \ a_1 = 7, \ a_2 = 21 \).

**Solution:** Since 2 is a double root of the characteristic polynomial we can’t use \( a_n = P2^n \) or even \( nP2^n \). We try \( a_n = n^2P2^n \). We have a mess:

\[
n^2P2^n = 7(n-1)^2P2^{n-1} - 16(n-2)^2P2^{n-2} + 12(n-3)^2P2^{n-3} - 2^n.
\]

Dividing by \( 2^{n-1} \), \( 2Pn^2 = 7P(n-1)^2P - 8P(n-2)^2 + 3P(n-3)^2 - 2 \), or

\[
P(2n^2 - 7n^2 + 14n - 7 + 8n^2 - 32n + 32 - 3n^2 + 18n - 27) = -2.
\]

This simplifies to \( -2P = -2 \), or \( P = 1 \). The fact that the \( n^2 \) terms and \( n \) terms cancelled tells us that our guess was right. The general solution is \( a_n = \)
\[(A+Bn)2^n + C3^n + 2^n,\] and the initial conditions give \(A+C = 4, 2A+2B+3C = 5, 4A+8B+9C = 5,\) leading to \(A = 3, B = -2, C = 1.\) The solution to the recurrence is \(a_n = 3^n + (n^2 - 2n + 3)2^n.\)

(e) \(a_n = 2a_{n-1} - a_{n-2}, a_0 = 1, a_1 = 3,\) USING generating functions.

Solution: We let \(A(x) = \sum_{n=0}^{\infty} a_n x^n.\) We pull off the first two terms and use the recurrence:

\[A(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} (2a_{n-1} - a_{n-2})x^n\]

We break the sum into two pieces \(2 \sum_{n=2}^{\infty} a_{n-1}x^n, -\sum_{n=2}^{\infty} a_{n-2}x^n.\) The second of these is \(-a_0x^2 - a_1x^3 - a_2x^4 + \cdots = -x^2A(x).\) For the first, we do something similar. Factoring an \(x\) out we have \(2x(a_1x + a_2x^2 + \cdots) = 2x(A(x) - a_0).\) Putting the pieces together, \(A(x) = a_0 + a_1 x + 2x(A(x) - a_0) - x^2A(x).\) We solve this equation for \(A(x)\) and we have \((x^2 - 2x + 1)A(x) = a_0 + x(a_1 - 2a_0).\) Plugging in the initial conditions,

\[A(x) = \frac{1 + x}{x^2 - 2x + 1} = \frac{1 + x}{(1-x)^2}.\]

\(a_n\) is the coefficient of \(x^n\) in \(A(x),\) or \(\binom{n+1}{1} + \binom{n}{1} = 2n + 1.\)

(f) \(a_n = 6a_{n-1} - 9a_{n-2} + 2 \cdot 3^n, a_0 = 1, a_1 = 0,\) with and without generating functions.

Solution: Without generating functions: The characteristic polynomial for the homogeneous problem is \(x^2 - 6x + 9 = (x - 3)^2,\) so the particular solution must have the form \(a_n = Pn^23^n.\) Plugging in,

\[Pn^23^n = 6P(n-1)^23^{n-1} - 9P(n-2)^23^{n-2} + 2 \cdot 3^n.\]

We divide by \(3^n\) giving \(Pn^2 = 2P(n-1)^2 - P(n-2)^2 + 2.\) So \(P(n^2 - 2n^2 + 4n - 2 + n^2 - 4n + 4) = 2\) or \(2P = 2.\) The general solution is \(a_n = (A + Bn)3^n + 3^n,\) and the initial conditions tell us \(A = 1, B = -2\) so \(a_n = (n^2 - 2n + 1)3^n.\)

With generating functions:

\[A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} (6a_{n-1} - 9a_{n-2} + 2 \cdot 3^n)x^n.\]
we have
\[\sum_{n=2}^{\infty} 6a_{n-1}x^n = 6x \sum_{n=2}^{\infty} a_{n-1}x^{n-1} = 6x(A(x) - a_0),\]
\[\sum_{n=2}^{\infty} 9a_{n-2}x^n = 9x^2 \sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 9x^2A(x)\]
and
\[\sum_{n=2}^{\infty} 2 \cdot 3^n x^n = 18x^2 \sum_{n=2}^{\infty} 3^{n-2}x^{n-2} = \frac{18x^2}{1 - 3x},\]
so
\[A(x) = a_0 + a_1x + 6x(A(x) - a_0) - 9x^2A(x) + \frac{18x^2}{1 - 3x},\]
or
\[(1 - 6x + 9x^2)A(x) = a_0 + a_1x - 6xa_0 + \frac{18x^2}{1 - 3x} = 1 - 6x + \frac{18x^2}{1 - 3x}.\]
Thus,
\[A(x) = \frac{1 - 6x}{(1 - 3x)^2} + \frac{18x^2}{(1 - 3x)^3}\]
and the coefficient of \(x^n\) is
\[\left(\frac{n+1}{1}\right)3^n - 6\left(\frac{n}{1}\right)3^{n-1} + 18\left(\frac{n}{2}\right)3^{n-2} = 3^n(n+1-2n+n^2-n) = (n^2-2n+1)3^n,\]
as before.

10. Solve each of the following recurrence relations.

(a) \(a_n = a_{n-1} + 4a_{n-2} + 4a_{n-3} + \cdots + 4a_0,\) \(a_0 = 1, a_1 = 10.\)

**Solution:** This can probably be done with generating functions but the following is easier. If \(n \geq 3\) then \(a_{n-1} = a_{n-2} + 4a_{n-3} + \cdots + 4a_0\) and if we subtract this from the original we have \(a_n - a_{n-1} = a_{n-1} + 3a_{n-2},\) or \(a_n = 2a_{n-1} + 3a_{n-2}.\) Now we have a second order recurrence, but only from \(n = 3\) on. It has characteristic polynomial \(x^2 - 2x - 3 = (x - 3)(x + 1).\) The general solution has the form \(a_n = A3^n + B(-1)^n.\) We can’t plug \(a_0\) and \(a_1\) into this, we need \(a_1\) and \(a_2\) instead. We use the original recurrence to tell us \(a_2 = a_1 + 4a_0 = 14.\) Now \(3A - B = 10, 9A + B = 14\) so \(A = 2, B = -4.\) Our solution is \(a_0 = 1, a_n = 2 \cdot 3^n - 4(-1)^n\) for \(n \geq 1.\) As a check, when \(n = 3\) we have \(a_3 = a_2 + 4a_1 + 4a_0 = 14 + 40 + 4 = 58\) and \(2 \cdot 3^3 - 4(-1)^3 = 54 + 4 = 58.\)

(b) \(a_n = a_{n-1} + 2a_{n-2} + 4a_{n-3} + \cdots + 2^{n-1}a_0,\) \(a_0 = 1.\)
Solution: This can be done fairly easily with generating functions because \(a_n\) is almost a convolution (of the \(a\)'s and \(2^n\)). We have

\[
A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} (a_{n-1} + 2a_{n-2} + 4a_{n-3} + \cdots + 2^{n-1}a_0)x^n.
\]

If we change the index on this last sum we have

\[
\sum_{n=0}^{\infty} (a_n + 2a_{n-1} + 4a_{n-2} + \cdots + 2^n a_0)x^{n+1} = x \sum_{n=0}^{\infty} (a_n + 2a_{n-1} + 4a_{n-2} + \cdots + 2^n a_0)x^n,
\]

which is a convolution (again, of \(a_0 + a_1 x + a_2 x^2 + \cdots\) and \(1 + 2x + 2^2 x^2 + \cdots\)). This gives \(A(x) = a_0 + xA(x)\frac{1}{1-2x} = 1 + A(x)\frac{x}{1-2x}\). Solving for \(A(x)\),

\[
\left(1 -\frac{x}{1-2x}\right) A(x) = 1, \quad \text{or} \quad \frac{1-3x}{1-2x} A(x) = 1.
\]

Thus, \(A(x) = \frac{1-2x}{1-3x}\), and \(a_n\) is the coefficient of \(x^n\) in this. That coefficient is 1 when \(n = 0\) and \(3^n - 2 \cdot 3^{n-1} = 3^{n-1}\) when \(n \geq 1\). So for \(n \geq 1\), \(a_n = 3^{n-1}\).

Easier is to use the same trick as in part (a). We can’t just subtract \(a_{n-1}\), in this case. When \(n \geq 2\) we can write \(a_{n-1} = a_{n-2} + 2a_{n-3} + \cdots + 2^{n-2} a_0\). If we multiply this by 2 and subtract from the original recurrence we have \(a_n - 2a_{n-1} = a_{n-1}\), or \(a_n = 3a_{n-1}\). Noting that \(a_1 = 1\), it follows that \(a_n = 3^{n-1}\) for all \(n \geq 1\).

Finally, I will mention that on this problem, it is easy to guess the answer. We have \(a_0 = 1, a_1 = a_0 = 1, a_2 = a_1 + 2a_0 = 3, a_3 = a_2 + 2a_1 + 4a_0 = 9\). If we guess \(a_0 = 1, a_n = 3^{n-1}\) for \(n \geq 1\) then

\[
a_n = a_{n-1} + 2a_{n-2} + \cdots + 2^{n-2} a_1 + 2^{n-1} a_0 = 3^{n-2} + 2 \cdot 3^{n-3} + 2^2 \cdot 3^{n-4} + \cdots + 2^{n-2} + 2^{n-1}.
\]

All but the last term are part of a geometric series with common ratio \(\frac{2}{3}\) so

\[
a_n = \frac{3^{n-2} - \frac{2}{3} 2^{n-2}}{1 - \frac{2}{3}} + 2^{n-1} = 3^{n-1} - 2^{n-1} + 2^{n-1} = 3^{n-1},
\]

verifying that the formula is correct.