1. (20 points) Find the appropriate generating function (in closed form) for each of the following problems. Do not find the coefficient of $x^n$.

(a) In how many ways can $n$ balls be put in 4 boxes if each box is nonempty?

\textbf{Solution:} The generating function for each box is $x + x^2 + x^3 + \cdots = \frac{x}{1-x}$.

The generating function for the problem is the fourth power of this, $\frac{x^4}{(1-x)^4}$.

(b) How many quaternary sequences (0’s, 1’s, 2’s, 3’s) of length $n$ are there having at least 1 of each of the four digits?

\textbf{Solution:} Here, the generating function is $(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)^4 = (e^x - 1)^4$.

(c) In how many ways can $n$ balls be put in 4 boxes if the first box has an even number of balls and the last box has an odd number of balls?

\textbf{Solution:} The generating function for the first box is $1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$.

For the last box, it is $x + x^3 + x^5 + \cdots = \frac{x}{1-x^2}$. For the problem we have $\frac{1}{1-x^2} \frac{1}{1-x} \frac{1}{1-x} \frac{x}{1-x^2} = \frac{x}{(1-x^2)^2(1-x)^2}$.

(d) How many quaternary sequences of length $n$ have an even number of 0’s and an odd number of 1’s?

\textbf{Solution:} Even number of 0’s: $\frac{e^x + e^{-x}}{2}$, odd number of 1’s: $\frac{e^x - e^{-x}}{2}$, our generating function is $\frac{e^x + e^{-x}}{2} \frac{e^x - e^{-x}}{2} e^{2x} = \frac{e^{4x} - 1}{4}$.
2. (20 points) Find the generating function for each of the following problems. Then SOLVE for the number asked.

(a) In how many ways can \(n\) balls be put into 4 boxes if the first box has at least 2 balls?

**Solution:** The generating function for the first box is \(x^2 + x^3 + x^4 + \cdots = \frac{x^2}{1 - x}\). For the problem, the generating function is \(\frac{x^2}{(1 - x)^4}\) and the coefficient of \(x^n\) in this is \(\binom{n - 2 + 4 - 1}{4 - 1} = \binom{n + 1}{3}\).

(b) How many quaternary sequences of length \(n\) have at least two 0’s?

**Solution:** For 0 we need \(x^2 + x^3 + \cdots = e^x - 1 - x\). The rest just contribute \(e^x\) each. The generating function for the problem is \(e^{4x} - e^{3x} - xe^{3x}\). The answer to the question is \(n!\) times the coefficient of \(x^n\) in this expression or
\[
n! \left(\frac{4^n}{n!} - \frac{3^n}{n!} - \frac{3^{n-1}}{(n-1)!}\right) = 4^n - 3^n - n3^{n-1}.
\]

3. (10 points) Find a recurrence, with initial conditions, for \(a_n\), the number of quaternary sequences of length \(n\) in which no odd number appears to the left of any even number. Hint: After the first even number appears, the remaining digits can only be 1’s and 3’s.

**Solution:** Note: either the statement of the problem is wrong or the hint is wrong. Let’s say the hint is wrong: it should read that after the first odd number, the remaining positions must consist of only odd numbers. If the sequence starts with either 0 or 2, then filling the remaining \(n - 1\) positions is the same kind of problem: they must have no odd number to the left of an even. The count for this is \(a_{n-1}\), so there are \(2a_{n-1}\) strings of this type. If the string starts with a 1 or a 3, then the remaining \(n - 1\) positions consist of only 1’s and 3’s, and there are a total of \(2 \cdot 2^{n-1}\) strings of this type. The total count is \(a_n = 2a_{n-1} + 2^n\).

For the initial conditions, the empty string fulfills the requirements so \(a_0 = 1\). Alternatively, all strings of length 1 are ok so \(a_1 = 4\).
4. (20 points) Solve each of the following recurrence relations

(a) \( a_n = 7a_{n-1} - 10a_{n-2}, \ a_0 = 1, \ a_1 = 11 \).

**Solution:** The characteristic polynomial is \( x^2 - 7x + 10 = (x - 2)(x - 5) \) so the general solution is \( a_n = A2^n + B5^n \). The initial conditions give us \( A + B = 1 \), \( 2A + 5B = 11 \). Multiplying the first by 2 and subtracting from the second, \( 3B = 9 \). Thus, \( B = 3 \), \( A = -2 \) and the solution is \( a_n = 3 \cdot 5^n - 2 \cdot 2^n = 3 \cdot 5^n - 2^{n+1} \).

(b) \( a_n = 6a_{n-1} - 9a_{n-2}, \ a_0 = 1, \ a_1 = 9 \).

**Solution:** Here, the characteristic polynomial is \( x^2 - 6x + 9 = (x - 3)^2 \). Since there is a multiple root the general solution is \( a_n = (An + B)3^n \). The initial conditions now say \( B = 1 \), \( 3A + 3B = 9 \), or \( A = 2 \). The solution is \( a_n = (2n + 1)3^n \).

5. (15 points) Consider the inhomogeneous recurrence relation \( a_n = 7a_{n-1} - 10a_{n-2} + f(n) \). You solved the homogeneous problem in part 4a.

(a) If \( f(n) = -2 \cdot 3^n \), find the particular solution to the problem.

**Solution:** From question 4, the characteristic polynomial for the homogeneous system is \( (x - 2)(x - 5) \). Since 3 is not a zero of this polynomial, for a particular solution, we should guess \( a_n = P3^n \). Trying this, \( P3^n = 7P3^{n-1} - 10P3^{n-2} - 2 \cdot 3^n \). Dividing by \( 3^{n-2} \), \( 9P = 21P - 10P - 18 \), or \( 2P = 18 \) so the particular solution is \( a_n = 9 \cdot 3^n = 3^{n+2} \).

(b) Solve the recurrence \( a_n = 7a_{n-1} - 10a_{n-2} - 2 \cdot 3^n, \ a_0 = 12, \ a_1 = 39 \).

**Solution:** The general solution to an inhomogeneous recurrence has the form (Particular solution to inhomogeneous recurrence) + (general homogeneous solution). We know the general homogeneous solution is \( A2^n + B5^n \) from question 4 so the general solution to the recurrence is \( A2^n + B5^n + \cdot 3^{n+2} \). The initial conditions yield \( A + B + 9 = 12 \), \( 2A + 5B + 27 = 39 \). From these, we get \( B = 2 \), \( A = 1 \) so the solution we seek is \( a_n = 2^n + 2 \cdot 5^n + 3^{n+2} \).

Since I did not indicate how one should solve the recurrence, there was no requirement that people use part (a). One could use generating functions, but that would require partial fractions. Instead, some people used the following: If we know that the particular solution has to have the form \( P3^n \) and the general solution will be \( a_n = A2^n + B5^n + (\text{particular}) = A2^n + B5^n + P3^n \), then we can...
create another initial condition and solve for $A, B, P$ using those. In this case, $a_2 = 135$. We have

$$ A + B + P = 12, \quad 2A + 5B + 3P = 39, \quad 4A + 25B + 9P = 135. $$

Subtracting twice the second from the third and then twice the first from the second, $15B + 3P = 57$, $3B + P = 15$, and we get $6B = 12 \rightarrow B = 2$, $P = 9$, $A = 1$.

(c) If $f(n) = 5 + 5 \cdot 8^n + n 5^n$, what should one guess for the particular solution? (Do not solve the undetermined coefficients problem).

**Solution:** One adds together (using different parameters), the particular solution guesses from $f(n) = 5$, $f(n) = 5 \cdot 8^n$ and $f(n) = n 5^n$. These particular solutions should be $P$ for $5$, $P \cdot 8^n$ for $5 \cdot 8^n$ and $(Pn^2 + Qn)5^n$ for $f(n) = n 5^n$. For the last case, one would normally guess $Pn^2 + Qn$ but $5^n$ is a solution to the homogeneous problem so an extra factor of $n$ is required. Changing the parameters and adding, we should guess $a_n = P_1 + P_2 8^n + (P_3 n^2 + P_4 n)5^n$.

6. (15 points) Use generating functions to solve each of the following recurrence relations.

(a) $a_n = 3a_{n-1} + 3^{n-1}$, $a_0 = 1$.

**Solution:** If we let $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$ then after the $n = 0$ term, we can use the recurrence. We have

$$ A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} (3a_{n-1} + 3^{n-1})x^n. $$

We break the sum into two pieces. For the first, $\sum_{n=1}^{\infty} 3a_{n-1}x^n = 3x \sum_{n=1}^{\infty} a_{n-1}x^{n-1} = 3xA(x)$. For the second piece, $\sum_{n=1}^{\infty} 3^{n-1}x^n = x \sum_{n=1}^{\infty} 3^{n-1}x^{n-1} = \frac{x}{1 - 3x}$. Putting the pieces together,

$$ A(x) = 1 + 3xA(x) + \frac{x}{1 - 3x} \quad \text{or} \quad (1 - 3x)A(x) = 1 + \frac{x}{1 - 3x} $$

so $A(x) = \frac{1}{1 - 3x} + \frac{x}{(1 - 3x)^2}$. Using the formula $\frac{1}{(1-ax)^m} = \sum_{n=0}^{\infty} \binom{m-1+n}{m-1} a^n x^n$, the coefficient of $x^n$ in this expression is $a_n = 3^n + n3^{n-1}$. 


(b) \( a_n = na_{n-1} + 3n - 3, \ a_0 = 1. \) Hint: You may use \( \sum_{n=1}^{\infty} (3n-3)\frac{x^n}{n!} = 3 - 3e^x(1-x). \)

**Solution:** This time \( A(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}. \) We have

\[
A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = a_0 + \sum_{n=1}^{\infty} a_n \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} (na_{n-1} + 3n - 3) \frac{x^n}{n!}.
\]

The first piece of the sum is
\[
\sum_{n=1}^{\infty} na_{n-1} \frac{x^n}{n!} = \sum_{n=1}^{\infty} a_{n-1} \frac{x^n}{(n-1)!} = x \sum_{n=1}^{\infty} a_{n-1} \frac{x^{n-1}}{(n-1)!} = xA(x)
\]

or \( xA(x) \). The hint tells how to sum the second piece. Thus,

\[
A(x) = 1 + xA(x) + 3 - 3e^x(1-x) \quad \text{or} \quad (1-x)A(x) = 4 - 3e^x(1-x)
\]

so \( A(x) = \frac{4}{1-x} - 3e^x \). The coefficient of \( x^n \) in \( A(x) \) is \( 4 + \frac{3}{n!} \) but we need to multiply this by \( n! \) (it’s an exponential GF) so the solution is \( a_n = 4n! - 3 \).

7. Some extra credit questions, if you are interested. Generating functions help on (d).

(a) (2 points) Give a direct count (that is, without using the recurrence) for \( a_n \) in problem 3.

**Solution:** The sequences described in problem 3 have the following form: the first \( k \) terms are 0 or 2 (even numbers), an odd number appears in position \( k+1 \) and the remaining positions are all 1 or 3 (odd numbers). It is possible that this transition never occurs, if all terms are even. This gives \( n \) possibilities for \( k \), with one more case if there are no odds. In every case, each position has two choices: one of two evens, or one of two odds. Thus, \( a_n = (n+1)2^n \).

(b) (8 points) Find, with initial conditions, a recurrence for \( a_n \), the number of sequences of length \( n \) made of 0’s, 1’s, 2’s, 3’s, 4’s, if all 0’s appear before any 2 and all 2’s appear before any 4. Solve the recurrence. I will give credit for sufficient progress.

(c) (4 points) Solve the recurrence \( a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3} + \cdots + na_0, \ a_0 = 1. \)

(d) (4 points) Solve the recurrence \( a_n = na_{n-1} + (n-1)a_{n-2} + \cdots + a_0, \ a_0 = 1. \)