1. Let \( A = \begin{pmatrix} 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 3 & 3 & 2 \\ 3 & 3 & 1 & 1 & -1 \end{pmatrix} \).

(a) Find the \( LU \) factorization of \( A \).

(b) Write \( L \) as a product of elementary matrices.

(c) Find all solutions to \( Ax = 0 \). Put your answer in vector parametric form.

(d) Use the \( LU \) decomposition for \( A \) to solve the system \( Ax = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix} \).

(e) Find all \( b \) for which \( Ax = b \) is consistent. Put your answer in vector parametric form. That is, write your answer in the form \( b = xv_1 + yv_2 \) for some \( v_1 \) and \( v_2 \).

2. Let \( A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{pmatrix} \).

(a) Find \( A^{-1} \) by row reduction.

(b) Find all cofactors of \( A \) and calculate the determinant of \( A \) all 6 ways by cofactors: across all rows, down all columns.

(c) Find the \( LU \) factorization of \( A \).

3. Suppose \( M = \begin{pmatrix} 0 & A \\ A & B \end{pmatrix} \) is a block matrix, and \( A \) and \( B \) are invertible.

(a) Show that \( M \) is invertible and \( M^{-1} = \begin{pmatrix} -A^{-1}BA^{-1} & A^{-1} \\ A^{-1} & 0 \end{pmatrix} \).

(b) If \( M \) is invertible, do \( A \) and \( B \) have to be invertible as well? Why or why not?

(c) Find a formula for the determinant of \( M \) in terms of the determinants of \( A \) and \( B \).

4. Find the determinants of each of the following matrices.

\[
\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 7 \end{pmatrix}.
\]

Can you find the general pattern? Can you find the general \( L \) and \( U \)?
5. In this problem we prove that \( \det(A^t) = \det(A) \). That is, a matrix and its transpose have the same determinant.

(a) Prove that the transpose of an elementary matrix is still an elementary matrix.

(b) Prove that if \( E \) is an elementary matrix, then \( \det(E^t) = \det(E) \).

(c) Use parts (a) and (b) to prove that for any square matrix \( A \), \( \det(A^t) = \det(A) \).

   Hint: Use the fact that \( (MN)^t = N^tM^t \) and the fact that the determinant of a product is the product of the determinants.

(d) Unrelated but still interesting, if \( A \) is a square matrix and \( E \) is an elementary matrix, then we know \( EA \) performs \( E \)'s row operation on \( A \). What does the matrix \( AE \) look like?

6. Let \( A \) and \( B \) be \( n \times n \) matrices.

(a) If \( A \) is not invertible, prove that \( AB \) is not invertible.

(b) If \( B \) is not invertible, prove that \( AB \) is not invertible.

Can you prove (a) and (b) in two different ways? One way should use determinants, the other should use problem 5 above.

7. Solve each of the following problems using linear algebra, that is, by solving systems of equations.

(a) Find the equation of the line \( ax + by + c = 0 \) passing through the points \((1, 2)\) and \((5, 8)\).

(b) Find the equation of a quadratic in the form \( x^2 = a + bx + cy \) passing through the points \((-1, -1), (1, 1), (3, 5)\).

(c) Show that there is no quadratic of the form \( x^2 = a + bx + cy \) passing the points \((-1, -1), (1, 1), (3, 3)\).

(d) Find a formula for the sum \( f(n) = 1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 \).