First, some comments. The first exam was almost entirely computational in nature. Only problems 1b and 5c involved more than straightforward memorized techniques, and we had done several problems like 1b already. This exam will still have its share of calculations, but it will have a much greater focus on ideas than the first.

Important concepts to know: What is a vector space/subspace? How do I test if something is a subspace? You should know that there are two standard approaches: We have the subspace test, and we also have the result that all spans are subspaces.

What is a span? What does it mean to be linearly independent?

What is a basis?

What are Rank and Nullity, and how do they relate to row reduction?

Related to all of these, there are computational methods as well. For example, how do you find a basis for a vector space? (Question 2) How do you find bases for the Null Space, Column Space and Row Space of a matrix? (Question 3). How do I test if a set is linearly independent? A Spanning set? (Question 5).

One final comment before getting on to the solutions for the review sheet. Systems of equations and matrices have been central to all of these ideas, but if a matrix, $A$, is related to a problem, you should explicitly know HOW it is related. For example, if you are given three vectors in $\mathbb{R}^4$ and you are asked if they are independent, you could form a matrix with the vectors as rows, or as columns. But to get credit on the question, you will have to show that you know why your specific matrix is related to the question, and how operations on that matrix answer the question. That is, if the vectors are $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 4 \\ 4 \end{pmatrix}$, then if you use the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}$, then the reason this matrix applies is that if $u, v, w$ are columns of a matrix, then a dependence relation $xu+yv+zw = 0$ corresponds to a nontrivial solution $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$, and we can check for nontrivial solutions by row reduction. On the other hand, if you instead used the matrix $B = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 4 & 4 \end{pmatrix}$, then the justification would be that row reduction does not change the row space, so if a row of zeros appears in the reduced matrix, then the rows of that matrix must be linearly dependent.
On to the solutions.

1. Which of the following are subspaces? In each case, justify your answer.
   
   (a) \( U = \) the set of all upper triangular matrices in \( M_{2 \times 2} \).

   **Solution:** Each of the spaces that are subspace can be shown to be either by the subspace test, or by showing they are spans. For this problem, \( U \) IS a subspace. If we used the subspace test, the demonstration looks like this: First, the zero matrix counts as upper triangular. Next, given two upper triangular matrices, \( A, B \), say
   
   \[
   A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad B = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix},
   \]
   
   then
   
   \[
   A + B = \begin{pmatrix} a + x & b + y \\ 0 & c + z \end{pmatrix},
   \]
   
   which is still upper triangular. Similarly, for any scalar, \( x \), \( xA = \begin{pmatrix} ax & bx \\ 0 & cx \end{pmatrix} \), an upper triangular matrix.

   Alternatively,
   
   \[
   A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
   \]
   
   so
   
   \[
   U = \text{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.
   \]

   (b) \( V = \) the set of all matrices in \( M_{2 \times 2} \) with at least one entry equal to zero.

   **Solution:** Things like this are not usually subspaces. To show something is not a subspace, you should give a specific example of one of the properties in the subspace test failing. That is, you should give an example with actual numbers. In this case, \( A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \) are both in \( V \), but their sum, \( A + B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) is not, so \( V \) is NOT a subspace.

   (c) \( W = \) the set of those polynomials in \( P_3 \) with coefficients of \( t \) and \( t^2 \) equal to each other. That is, \( 2t^3 - t^2 - t + 5 \) is in \( W \) but \( 4t^3 + 3t^2 + 2t + 1 \) is not.

   **Solution:** \( W \) IS a subspace. If we were to use the subspace test, the zero polynomial is in \( W \): it has 0 as the coefficients of \( t \) and \( t^2 \). If \( p(t) \) and \( q(t) \) are in \( W \), then \( p(t) = at^3 + bt^2 + bt + c \) and \( q(t) = dt^3 + et^2 + et + f \) for some scalars \( a, b, c, d, e, f \). We have
   
   \[
   p(t) + q(t) = (a + d)t^3 + (b + e)t^2 + (b + e)t + (c + f),
   \]
   
   which has equal coefficients for \( t, t^2 \), so \( p(t) + q(t) \) is in \( W \). Similarly, \( kp(t) = kat^3 + kbt^2 + kbt + kc \) has equal coefficients for \( t, t^2 \), so \( kp(t) \) is also in \( W \).

   If we wanted to use the spanning property instead, then
   
   \[
   p(t) = at^3 + bt^2 + bt + c = at^3 + b(t^2 + t) + c \cdot 1,
   \]
   
   so \( W = \text{Span}\{t^3, t^2 + t, 1\} \).
(d) $X$, the subset of $P_3$ with $p(1) \neq p(-1)$.

**Solution:** This $X$ is NOT a subspace. The simplest demonstration is that it does not contain the zero vector. That is, if $p(t)$ is the zero polynomial, then $p(1) = 0 = p(-1)$ so $p(t)$ is not in $U$.

2. Find bases for each of the following subspaces.

(a) $U$ = the set of all points on the plane $x - y - 2z = 0$.

**Solution:** If we view this as a single equation in three unknowns, with matrix $(1 \ -1 \ -2)$, then we can take $x$ to be the dependent variable and $y, z$ free. We have

$$
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
y + 2z \\
y \\
z
\end{pmatrix} =
y \begin{pmatrix} 1 \\
2 \\
1
\end{pmatrix} + z \begin{pmatrix} 0 \\
1 \\
2
\end{pmatrix},
$$

so a basis is

$$
\left\{ \begin{pmatrix} 1 \\
2 \\
0
\end{pmatrix}, \begin{pmatrix} 1 \\
0 \\
0
\end{pmatrix} \right\}.
$$

(b) $V$ = the set of all $2 \times 2$ matrices $A$ with $A = A^t$.

**Solution:** We need

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} =
\begin{pmatrix}
a & c \\
b & d
\end{pmatrix}
$$

so $V$ is the set of all matrices of the form

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
$$

We write in terms of $a, b, d$: $\begin{pmatrix} a & b \\
b & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\
0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix} +
\begin{pmatrix} 0 & 0 \\
0 & 1 \end{pmatrix},
$$

so a basis is

$$
\left\{ \begin{pmatrix} 1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix} 0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix} 0 \\
0 \\
1
\end{pmatrix} \right\}.
$$

(c) $W$ = the set of all polynomials in $P_3$ with $p(1) = p(-1)$.

**Solution:** The general polynomial in $P_3$ has the form $at^3 + bt^2 + ct + d$. To be in $W$, we need $a + b + c + d = -a + b - c + d$, or $2a + 2c = 0$. One condition on a 4-dimensional space means that $W$ should be 3-dimensional. If we say the determined variable is $c$, then $a, b, d$ are free. We have

$$
p(t) = at^3 + bt^2 - at + d = a(t^3 - t) + bt^2 + d \cdot 1,
$$

so a basis is

$$
\{t^3 - t, t^2, 1\}.
$$
3. Let $A$ be the matrix \[
\begin{pmatrix}
1 & 1 & 1 & 3 & 2 \\
3 & 3 & 5 & 5 & 12 \\
5 & 5 & -1 & 27 & -8
\end{pmatrix}.
\]

(a) Find a basis for the Null Space of $A$.

**Solution:** A first step for most of these problems is to row reduce $A$. We have
\[
\begin{pmatrix}
1 & 1 & 1 & 3 & 2 \\
3 & 3 & 5 & 5 & 12 \\
5 & 5 & -1 & 27 & -8
\end{pmatrix} \Rightarrow
\begin{pmatrix}
1 & 1 & 1 & 3 & 2 \\
0 & 0 & 2 & -4 & 6 \\
0 & 0 & -6 & 12 & -18
\end{pmatrix} \Rightarrow
\begin{pmatrix}
1 & 1 & 0 & 5 & -1 \\
0 & 0 & 1 & -2 & 3 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
If the variables are $x_1, \ldots, x_5$, then $x_1 = -x_2 - 5x_4 + x_5$, $x_3 = 2x_4 - 3x_5$, so
\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix} = \begin{pmatrix}
x_2 - 5x_4 + x_5 \\
x_2 \\
2x_4 - 3x_5 \\
x_4 \\
x_5
\end{pmatrix} = \begin{pmatrix}
-1 \\
1 \\
0 \\
0 \\
0
\end{pmatrix} x_2 + \begin{pmatrix}
-5 \\
0 \\
2 \\
1 \\
0
\end{pmatrix} x_4 + \begin{pmatrix}
1 \\
0 \\
-3 \\
0 \\
1
\end{pmatrix} x_5,
\]
and a basis consists of the three vectors multiplying $x_2, x_4, x_5$.

(b) Find a basis for the Column Space of $A$.

**Solution:** Looking at the reduced matrix, we see the first and third columns are the pivot columns so a basis for $\text{Col } A$ is \[
\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \\ 1 \\ -1 \\ 3 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \\ 1 \\ -1 \\ 3 \\ 5 \\ -1 \end{pmatrix} \right\}.
\]

(c) Find a basis for the Row Space of $A$.

**Solution:** We take the nonzero rows of the reduced matrix for the basis, so the basis is \{ $(1, 1, 0, 5, -1)$, $(0, 0, 1, -2, 3)$ \}.

(d) By row reducing $A^t$, find a “nice” basis for the Column Space of $A$.

**Solution:** We could row reduce all of $A^t$ but it makes sense to only reduce the independent columns. This row reduction would proceed \[
\begin{pmatrix}
1 & 3 & 5 \\
0 & 2 & -6
\end{pmatrix} \Rightarrow \begin{pmatrix}
1 & 0 & 14 \\
0 & 1 & -3
\end{pmatrix}.
\]
Converting back to columns, the basis is \[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 14 \\ 14 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.
\]
(e) Write the fourth column of \( A \) as a linear combinations of earlier columns.

**Solution:** The easiest way to do this is to look at the reduced matrix since the relationships among the columns are the same for \( A \) and its reduced matrix. We see that the fourth column of the reduced matrix is \( 5(\text{col } 1) - 2(\text{col } 3) \). This means that the same is true for \( A \):

\[
\begin{pmatrix} 3 \\ 5 \\ 27 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}.
\]

(f) Find another dependence relation among the columns of \( A \) (other than column \( 1 = \text{column } 2 \)).

**Solution:** Write the fifth column as a combination of columns 1, 3. Again, based on the reduced matrix we have

\[
\begin{pmatrix} 2 \\ 12 \\ -8 \end{pmatrix} = - \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}.
\]

(g) Find a dependence relation among the rows of \( A \).

**Solution:** Looking at the reduced matrix in part (d), \( R = \begin{pmatrix} 1 & 0 & 14 \\ 0 & 1 & -3 \end{pmatrix} \), we might guess that the third row is \( 14(\text{row } 1) - 3(\text{row } 2) \). In fact, this is true.

4. Suppose \( \{u, v, w\} \) is a linearly independent set.

(a) Show that \( \{u + v, v + 2w, u + 2v + 2w\} \) is linearly dependent.

**Solution:** I have an instinctive reaction to problems like this: Set a general linear combination equal to 0 and see if all variables are forced to be zero. Here, we have \( x(u + v) + y(v + 2w) + z(u + 2v + 2w) = 0 \). This leads to the equations \( x + z = 0, x + y + 2z = 0, 2y + 2z = 0 \). I got these equations by looking at what multiplies \( u \), what multiplies \( v \), and what multiplies \( w \). A quick row reduction:

\[
\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

If we pick \( z = -1 \), this says \( x = 1, y = 1 \). In fact, we can check that \( (u + v) + (v + 2w) - (u + 2v + 2w) = 0 \). It is fine to just notice this linear dependence (and state it) without solving a system of equations.
(b) Carefully show that $4u + 6v + 4w$ is in $\text{Span}\{u + v, v + 2w, u + 2v + 2w\}$. Explain your reasoning.

**Solution:** We just exhibit the combination that works. If it is easy to spot, just give the combination. Otherwise, you can always solve a system of equations: Write $x(u + v) + y(v + 2w) + z(u + 2v + 2w) = 4u + 6v + 4w$, from which we could say we want $x + z = 4$, $x + y + 2z = 6$, $2y + 2z = 4$. If we assume there is a solution, and as before, $z$ will be free, then we could set $z = 0$ to get $y = 2, x = 4$ so $4(u + v) + 2(v + 2w) + 0(u + 2v + 2w) = 4u + 6v + 4w$, as desired.

(c) Carefully show that $6u + 4v + 6w$ is not in $\text{Span}\{u + v, v + 2w, u + 2v + 2w\}$. Explain your reasoning.

**Solution:** This is harder than part (b). We start the same way: Write $x(u + v) + y(v + 2w) + z(u + 2v + 2w) = 6u + 4v + 6w$, but how do we know that this means $x + z = 6$, $x + y + 2z = 4$, $2y + 2z = 6$? In part (b), any $x, y, z$ which satisfied the desired equations provided a demonstration that $4u + 6v + 4w$ was in the span. But note that linear independence was never invoked in part (b). In fact, it was not needed. Here, independence is critical: if $u = v = w$ then $6u + 4v + 6w$ IS in $\text{Span}\{u + v, v + 2w, u + 2v + 2w\}$. How does independence come into play? If we rewrite $x(u + v) + y(v + 2w) + z(u + 2v + 2w) = 6u + 4v + 6w$ in the form $u(x + z - 6) + v(x + y + 2z - 4) + w(2y + 2z - 6) = 0$ then the independence of $\{u, v, w\}$ forces $x + z = 6$, $x + y + 2z = 4$, $2y + 2z = 6$. That is, without independence, these equations could be true but are not forced on us. With independence, we must have these equations. Now, we show the equations are inconsistent, and this shows $6u + 4v + 6w$ is not in the span. To show the equations are inconsistent, the second minus the first is $y + z = 0$, or $y + z = -2$. However, the third can be rewritten $y + z = 3$, and we can’t have $y + z$ equaling both -2 and 3.

5. Extend the set $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ to a basis for $M_{2\times 2}$. Explain your reasoning.

**Solution:** One way to do this is to append the standard basis to this set. A spanning set for $M_{2\times 2}$ is

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$ 

Call these vectors $v_1, v_2, v_3, v_4, v_5, v_6$, respectively. Then by inspection we see that $v_1 - v_3 - v_4 = 0$ and $v_2 - v_5 - v_6 = 0$. That is, we can write $v_4 = v_1 - v_3$ and
\( v_6 = v_2 - v_5 \). These dependencies mean we can remove \( v_4 \) and \( v_6 \) from the set without changing its span. Thus,

\[
\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}
\]

is still a spanning set. Because this set has four vectors and \( \text{dim}(M_{2 \times 2}) = 4 \), this set is a basis.

6. The matrix \( A \) in problem 3 was just one example of a \( 3 \times 5 \) matrix; there are lots more \( 3 \times 5 \) matrices around. Fill in the blanks: For every \( 3 \times 5 \) matrix \( B \),

\[ \underline{\quad} \leq \text{dim}(\text{Null}(B)) \leq \underline{\quad}. \]

**Solution:** The answer is \( 2 \leq \text{dim}(\text{Null}(B)) \leq 5 \).

Justify your answer.

**Solution:** We use the Rank-Nullity theorem: The rank of a matrix + its nullity = the number of columns. Also, the rank is the number of pivots. Putting these together, \( 5 - \text{number of pivots in } B = \text{dim}(\text{Null}(B)) \). Since \( B \) has three rows, it can have at most 3 pivots, so \( 5 - \text{number of pivots in } B \geq 2 \). The other extreme is where \( B \) has no pivots (in this case, \( B \) is the zero matrix), which has \( 5 - \text{number of pivots in } B = 5 \).

7. Suppose \( A \) is a \( 3 \times 3 \) matrix and \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \) are both in \( \text{Nul}(A) \).

(a) Show that \( \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \) is in \( \text{Nul}(A) \).

**Solution:** If \( v \) is in the null space of \( A \), then \( Av = 0 \). Since \( A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \) and \( A \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0 \), we have \( A \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = A \left( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right) = A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + A \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \)
0 + 0 = 0, so \[
\begin{pmatrix}
2 \\
3 \\
2
\end{pmatrix}
\] is in the null space.

Alternatively, the Null Space is a vector space, and since vector spaces are closed under addition, \[
\begin{pmatrix}
2 \\
3 \\
2
\end{pmatrix}
\] is in the null space.

(b) What can you say about the rank and nullity of \( A \)?

**Solution:** Since \[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\] and \[
\begin{pmatrix}
1 \\
2 \\
1
\end{pmatrix}
\] are in \( \text{Nul}(A) \) and these vectors are independent, it must be that the nullity of \( A \) is at least 2. It IS possible that the nullity of \( A \) could be 3, the 0-matrix has nullity 3. Since \( \text{Rank} + \text{Nullity} = \) number of columns = 3 here, if Nullity = 3 then Rank = 0. If Nullity = 2, then Rank = 1.