1. (20 points) Which of the following are subspaces of $P_2$? Justify your answers.

(a) $U$, the set of those polynomials with constant term equal to the coefficient of $t^2$. For example, $-5t^2 - 5t - 5$ and $7t = 0t^2 + 7t + 0$ are in $U$.

**Solution:** Simplest is that (some justification needed) $U = \text{Span}\{t^2 + 1, t\}$ and all spans are subspaces. We could also use the subspace test, as most people in the class did. Here, the solution would be as follows: $U$ is clearly not empty. Suppose that $p(t)$ and $q(t)$ are in $U$, say $p(t) = at^2 + bt + a$ and $q(t) = ct^2 + dt + c$. Then $p(t) + q(t) = (a + c)t^2 + (b + d)t + (q + c)$, which has constant term equal to its coefficient of $t^2$ so $p(t) + q(t)$ is in $U$. Also, $kp(t) = kat^2 + kbt + ka$ again has constant term equal to its coefficient of $t^2$ so $kp(t)$ is in $U$. Thus, $U$ is nonempty and closed under vector addition and scalar multiplication so $U$ **IS** a subspace of $P_2$.

(b) $V$, the set of those polynomials with constant term not equal to coefficient of $t^2$. For example, $2t^2 + 3t + 3$ is in $V$ (because $2 \neq 3$).

**Solution:** $V$ is **NOT** a subspace of $P_2$ because it does not contain the 0-vector. As an alternative, $2t^2 + 3$ and $3t^2 + 2$ are in $V$, but their sum, $5t^2 + 5$ is not, so $V$ is not closed under vector addition.

(c) $W$, the set of those polynomials in which at least two coefficients are the same. For example, $-5t^2 - 5t - 5$ and $2t^2 + 3t + 3$ are in $W$.

**Solution:** $W$ is also **NOT** a subspace of $P_2$, again because of failure to be closed under vector addition. An example might be that $t^2 + 2t + 2$ and $t^2 + t + 2$ are both in $W$, but their sum, $2t^2 + 3t + 4$ is not.
2. (20 points) Find bases for the following vector spaces.

(a) \( V = \{at^3 + (a + b)t^2 + (a - b)t + b \mid a, b \in \mathbb{R}\} \). \( V \) is a subspace of \( P_3 \).

**Solution:** Since \( at^3 + (a + b)t^2 + (a - b)t + b = a(t^3 + t^2 + t) + b(t^2 - t + 1) \), it is clear that \( V = \text{Span}\{t^3 + t^2 + t, t^2 - t + 1\} \). Neither vector is a multiple of the other, so this set is also a basis.

(b) \( W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b = a + c + d \right\} \). \( W \) is a subspace of \( M_{2 \times 2} \).

**Solution:** The general object in \( W \) has the form

\[
\begin{pmatrix} a & a + c + d \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}
\]

so a spanning set is \( \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} \). These vectors are also linearly independent: If \( a \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a + c + d \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), then we need \( a = c = d = 0 \). Thus, this set is also a basis.

3. (10 points) Suppose that \( \{u, v, w\} \) is a linearly independent set.

(a) Prove that \( \{u + v + w, v + w, w\} \) is also linearly independent.

**Solution:** Suppose that \( xu + v + w + y(v + w) + zw = 0 \). We write this as \( xu + (x + y)v + (x + y + z)w = 0 \). We have a combination of \( u, v, w \) equal to 0, but \( \{u, v, w\} \) is linearly independent. This means that the coefficients of this combination must be 0. That is, we must have \( x = 0, x + y = 0, x + y + z = 0 \). The first equation, with the second force \( y = 0 \), and these with the third equation force \( z = 0 \). Thus, \( x, y, z \) must be 0, so \( \{u + v + w, v + w, w\} \) is linearly independent.

(b) However, if \( w = 2u + 2v \), show that \( \{u + v + w, v + w, w\} \) is linearly dependent.

**Solution:** In this case, \( \{u + v + w, v + w, w\} = \{3u + 3v, 2u + 3v, 2u + 2v\} \). Since \( 2(3u + 3v) = 3(2u + 2v) \) we have a dependence relation. Thus, this set is dependent.
4. (30 points) Let $A$ be the matrix \[
\begin{pmatrix}
1 & 1 & 2 & 3 & 2 \\
3 & 3 & 6 & 9 & 6 \\
5 & 5 & 7 & 9 & 7 
\end{pmatrix}.
\]

(a) Find a basis for the Null Space of $A$.

**Solution:** As a first step, we row reduce:
\[
\begin{pmatrix}
1 & 1 & 2 & 3 & 2 \\
3 & 3 & 6 & 9 & 6 \\
5 & 5 & 7 & 9 & 7 
\end{pmatrix} \Rightarrow \begin{pmatrix}
1 & 1 & 2 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & -6 & -3 
\end{pmatrix} \Rightarrow \begin{pmatrix}
1 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 
\end{pmatrix}
\]

so $x_1 = -x_2 + x_4$, $x_3 = -2x_4 - x_5$, $\vec{x} = \begin{pmatrix} -x_2 + x_4 \\ x_2 \\ -2x_4 - x_5 \\ x_4 \\ x_5 \end{pmatrix}$, and this leads to the basis
\[
\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}.
\]

(b) Find **two different** bases for the Column Space of $A$. One should be “nice”.

**Solution:** The pivot columns of $A$ form a basis, giving $\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$. If we change these to rows and reduce:
\[
\begin{pmatrix} 1 & 3 & 5 \\ 2 & 6 & 7 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & 0 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Going back to columns, a nice basis is $\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(c) Find a basis for the row space of $A$.

**Solution:** The nonzero rows of the reduced matrix form a basis for the row space of $A$ so a basis is $\{(1, 1, 0, -1, 0), (0, 0, 1, 2, 1)\}$. 
5. (10 points) Let \( T = \{t^3 + t^2 + t + 1, 2t^3 + 2t^2 + t + 1, 3t^3 + 3t^2 + t + 1, 3t^3 + 2t^2 + 2t + 1\} \).

(a) Find a dependence relationship among the vectors of \( T \).

**Solution:** Let’s call the polynomials \( p_1(t), p_2(t), p_3(t), p_4(t) \). Some people noted by inspection that \( p_3(t) = 2p_2(t) - p_1(t) \). To spot this dependence, we could write \( ap_1(t) + bp_2(t) + cp_3(t) + dp_4(t) = 0 \) and look for solutions \( a, b, c, d \) and note that

\[
\begin{align*}
 a(t^3 + t^2 + t + 1) + b(2t^3 + 2t^2 + t + 1) + c(3t^3 + 3t^2 + t + 1) + d(3t^3 + 2t^2 + 2t + 1) \\
= (a + 2b + 3c + 3d)t^3 + (a + 2b + 3c + 2d)t^2 + (a + b + c + 2d)t + (a + b + c + d).
\end{align*}
\]

For this to be 0, we need each coefficient to be 0. That is,

\[
\begin{align*}
 a + 2b + 3c + 3d &= 0 \\
 a + 2b + 3c + 2d &= 0 \\
 a + b + c + 2d &= 0 \\
 a + b + c + d &= 0
\end{align*}
\]

\[
\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

This tells us that we need \( d = 0 \) but \( c \) is free. Setting it equal to 1, \( a = 1, b = -2, c = 1 \), which gets us back to the dependence given.

(b) Find a basis for the subspace spanned by \( T \). Justify your answer.

**Solution:** Removing a dependent vector does not change the span, so \( \text{Span}(T) = \text{Span}\{p_1, p_2, p_4\} \), where we have removed \( p_3 \) from the set. I claim these polynomials are independent, so they form an independent spanning set, that is, a basis. To check independence, we ask that \( ap_1 + bp_2 + cp_4 = 0 \) and prove that \( a, b, c \) are forced to be 0. This is the same as setting \( c = 0 \) in part (a) of the problem, or deleting the third column of the matrix. When we do, the resulting reduced matrix will have a pivot in each column, meaning we need \( a = 0, b = 0, c = 0 \), as desired.
6. (10 points) Let \( U = \{ t^3, \ t^3 + t^2 + t + 1, \ 2t^3 + 2t^2 + t + 1, \ 3t^3 + 3t^2 + t + 1, \ 3t^3 + 2t^2 + 2t + 1 \} \). That is, \( U \) is the same as \( T \) in problem 5, but with \( t^3 \) included as well.

(a) Show that the span of \( U \) is all of \( P_3 \). One way: show the general polynomial is a combination of the polynomials in \( U \).

Solution: Following the hint, we try to write \( at^3 + bt^2 + ct + d \) as a combination of the five polynomials given. If we write

\[
\begin{align*}
    at^3 + bt^2 + ct + d &= x_1t^3 + x_2(t^3 + t^2 + t + 1) + x_3(2t^3 + 2t^2 + t + 1) \\
    &\quad\quad\quad\quad\quad\quad\quad\quad\quad+ x_4(3t^3 + 3t^2 + t + 1) + x_5(3t^3 + 2t^2 + 2t + 1) \\
    x_1 + x_2 + 2x_3 + 3x_4 + 3x_5 &= a \\
    x_2 + 2x_3 + 3x_4 + 2x_5 &= b \\
    x_2 + x_3 + x_4 + 2x_5 &= c \\
    x_2 + x_3 + x_4 + x_5 &= d
\end{align*}
\]

then matching coefficients, we need

\[
\begin{pmatrix}
    1 & 1 & 2 & 3 & 3 & a \\
    0 & 1 & 2 & 3 & 2 & b \\
    0 & 1 & 1 & 1 & 2 & c \\
    0 & 1 & 1 & 1 & 1 & d
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
    1 & 1 & 2 & 3 & 3 & a \\
    0 & 1 & 1 & 1 & 1 & d \\
    0 & 0 & 1 & 2 & 1 & b - d \\
    0 & 0 & 0 & 0 & 1 & c - d
\end{pmatrix}
\]

Since there is a pivot in every row, this system is ALWAYS consistent, meaning that no matter what \( a, b, c, d \) we choose, we will always be able to find the associated \( x_1, \ldots, x_5 \). That is, all polynomials in \( P_3 \) are combinations of the vectors in \( U \), or \( \text{Span}(U) = P_3 \).

(b) Find a subset of \( U \) that forms a basis for \( P_3 \). Justify your answer.

Solution: As we noted in problem 5 (a), \( 3t^3 + 3t^2 + t + 1 \) is a combination of the two previous polynomials in \( U \). Thus, we can delete it from \( U \) without changing the span. Doing this, we have four remaining polynomials, and the span, \( P_3 \), is four-dimensional, the the remaining vectors must be a basis for \( P_3 \). That is, a basis is \( \{ t^3, \ t^3 + t^2 + t + 1, \ 2t^3 + 2t^2 + t + 1, \ 3t^3 + 2t^2 + 2t + 1 \} \).
Some **extra credit** you can think about if you have time.

7. (15 points) This problem deals with subspaces of $M_{2\times 3}$. Suppose we define the reverse of a matrix by $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}^R = \begin{pmatrix} c & b & a \\ f & e & d \end{pmatrix}$ and the flip by $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}^F = \begin{pmatrix} d & e & f \\ a & b & c \end{pmatrix}$.

For example, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^R = \begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^F = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$.

(a) Let $V = \{ A \in M_{2\times 3} \mid A = A^R \}$. Show that $V$ is a subspace of $M_{2\times 3}$ and find a basis for $V$.

(b) Let $W = \{ A \in M_{2\times 3} \mid A = A^F \}$. Find a basis for $W$.

(c) Find a basis for $V \cap W$.

(d) Redo parts a, b, c when we replace $M_{2\times 3}$ by $M_{5\times 5}$.