1. Two bases for $M_{2\times 2}$ are 

$$B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \quad \text{and} \quad C = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$ 

(a) Find the transition matrix $P_{B\leftarrow C}$.

(b) Use part (a) to find $[A]_B$ given that $[A]_C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

2. Let $V$ be the subspace of $P_3$ consisting of all polynomials $p(t)$ with $p(3) = 0$. Here are two bases for $V$:

$$B = \{t - 3, (t - 3)^2, (t - 3)^3\}, \quad C = \{t^3 - 3t^2, t^2 - 3t, t - 3\}.$$ 

(a) Find the transition matrix $P_{B\leftarrow C}$.

(b) Find the transition matrix $P_{C\leftarrow B}$.

(c) Find $[p(t)]_C$, if $[p(t)]_B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

3. Some problems related to rotations, projections, and reflections.

(a) Find the matrix for the orthogonal projection of $R^2$ onto the line $y = 3x$.

(b) Find the matrix for the orthogonal reflection of $R^2$ through the line $y = 3x$.

(c) Find the matrix for the orthogonal projection of $R^2$ onto the line $y = \frac{1}{3}x$.

(d) Find the matrix for the orthogonal reflection of $R^2$ through the line $y = \frac{1}{3}x$.

(e) There is a result that says if you take the point $(x, y)$, reflect it through one line, and reflect that through another line, you will get a rotation. Suppose you take $(x, y)$, reflect through $y = \frac{1}{3}x$ and then that through $y = 3x$.

i. Find the matrix for the transformation that does this. (It should be the product of the matrices in parts b, d, but in what order?)

ii. Show that this matrix is a rotation matrix.

4. For each function $T$, show that $T$ is linear, and find bases for the kernel and range.

(a) $T : M_{2\times 2} \rightarrow M_{2\times 2}$ defined by $T(A) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} A - A \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

(b) $T : P_2 \rightarrow M_{2\times 2}$ defined by $T(at^2 + bt + c) = a \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} + b \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + cI$. 
5. Define a transformation \( T : R^4 \to R^3 \) as follows: Given a point \((w, x, y, z)\), map it to \((a, b, c)\) where \(a\) is the average of the first two coordinates, \(b\) is the average of the next two, and \(c\) is the average of the last two. For example, \(T(3, 5, 6, 8) = \left(4, \frac{11}{2}, 7\right)\).

(a) Find \(T(e)\) for each standard basis vector, \(e\).

(b) Find the matrix of \(T\).

6. Let \(T : P_2 \to P_2\) be a linear transformation for which
\[
T(t^2 + t + 1) = t + 1, \quad T(t + 1) = 0, \quad T(1) = t^2 + t + 1.
\]

(a) Find a formula for \(T(at^2 + bt + c)\). Hint: First write \(at^2 + bt + c\) as a combination of \(t^2 + t + 1, t + 1, 1\).

(b) What can you say about the dimension of the range of \(T\) and the dimension of the kernel of \(T\)? Note: This question could be answered independent of part (a), or by using part (a).

7. (a) Is there a linear transformation \(T : M_{2\times2} \to M_{2\times2}\) for which BOTH the kernel and the range are spanned by the set \(\left\{\begin{pmatrix}1 & 1 \\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 \\ 1 & 1\end{pmatrix}\right\}\)? If so, find a formula for such a transformation. If not, prove that no such transformation exists.

(b) Is there a linear transformation \(T : M_{2\times2} \to M_{2\times2}\) for which BOTH the kernel and the range are spanned by the set \(\left\{\begin{pmatrix}1 & 1 \\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 \\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 \\ 1 & 0\end{pmatrix}\right\}\)? If so, find a formula for such a transformation. If not, prove that no such transformation exists.

8. Find all eigenvalues and a basis for each eigenspace for the matrices:

\[
(a) \ A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \quad (b) \ A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad (c) \ A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \quad (d) \ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}
\]

9. (a) If \(v\) is an eigenvector for a matrix \(A\), show that \(v\) is also an eigenvector for \(A^2\).

(b) If \(c\) is an eigenvalue of \(A\), show that \(c^2\) is an eigenvector for \(A^2\).

10. Suppose that \(A\) is a matrix for which \(A^3 = 0\) but \(A^2 \neq 0\). Let \(u\) be a vector for which \(A^2u \neq 0\). Prove that the set \(\{u, Au, A^2u\}\) is linearly independent. Hint: Form a linear combination, set it equal to 0, and multiply by \(A\) and by \(A^2\) to get more relationships.

11. Give an example of a \(3 \times 3\) matrix \(A\) for which \(A^3 = 0\) but \(A^2 \neq 0\).