1. Two bases for $M_{2 \times 2}$ are 

$$B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\} \quad \text{and} \quad C = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$ 

(a) Find the transition matrix $P_{B \to C}$.

(b) Use part (a) to find $[A]_B$ given that $[A]_C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

2. Let $V$ be the subspace of $P_3$ consisting of all polynomials $p(t)$ with $p(3) = 0$. Here are two bases for $V$:

$$B = \{ t - 3, (t - 3)^2, (t - 3)^3 \}, \quad C = \{ t^3 - 3t^2, t^2 - 3t, t - 3 \}.$$ 

(a) Find the transition matrix $P_{B \to C}$.

(b) Find the transition matrix $P_{C \to B}$.

(c) Find $[p(t)]_C$, if $[p(t)]_B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

3. For each function $T$, show that $T$ is linear, and find bases for the kernel and range.

(a) $T : M_{2 \times 2} \to M_{2 \times 2}$ defined by $T(A) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} A - A \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

(b) $T : P_2 \to M_{2 \times 2}$ defined by $T(at^2 + bt + c) = a \begin{pmatrix} 5 \\ 4 \\ 5 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + cI$.

4. Define a transformation $T : R^4 \to R^3$ as follows: Given a point $(w, x, y, z)$, map it to $(a, b, c)$ where $a$ is the average of the first two coordinates, $b$ is the average of the next two, and $c$ is the average of the last two. For example, $T(3, 5, 6, 8) = (4, \frac{11}{2}, 7)$.

(a) Find $T(e)$ for each standard basis vector, $e$.

(b) Find the matrix of $T$. 
5. Let $T : P_2 \rightarrow P_2$ be a linear transformation for which

$T(t^2 + t + 1) = t + 1, \quad T(t + 1) = 0, \quad T(1) = t^2 + t + 1.$

(a) Find a formula for $T(at^2 + bt + c)$. Hint: First write $at^2 + bt + c$ as a combination of $t^2 + t + 1, t + 1, 1$.

(b) What can you say about the dimension of the range of $T$ and the dimension of the kernel of $T$? Note: This question could be answered independent of part (a), or by using part (a).

6. (a) Is there a linear transformation $T : M_{2\times2} \rightarrow M_{2\times2}$ for which BOTH the kernel and the range are spanned by the set $\{(1 1 0 0), (0 0 1 1)\}$? If so, find a formula for such a transformation. If not, prove that no such transformation exists.

(b) Is there a linear transformation $T : M_{2\times2} \rightarrow M_{2\times2}$ for which BOTH the kernel and the range are spanned by the set $\{(1 1 0 0), (0 0 1 1), (1 0 1 0)\}$? If so, find a formula for such a transformation. If not, prove that no such transformation exists.

7. Find all eigenvalues and a basis for each eigenspace.

(a) $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$  
(b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$
(c) $A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix}$
(d) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

8. (a) If $v$ is an eigenvector for a matrix $A$, show that $v$ is also an eigenvector for $A^2$.

(b) If $c$ is an eigenvalue of $A$, show that $c^2$ is an eigenvalue for $A^2$.

9. Suppose that $A$ is a matrix for which $A^3 = 0$ but $A^2 \neq 0$. Let $u$ be a vector for which $A^2u \neq 0$. Prove that the set $\{u, Au, A^2u\}$ is linearly independent. Hint: Form a linear combination, set it equal to 0, and multiply by $A$ and by $A^2$ to get more relationships.

10. Give an example of a $3 \times 3$ matrix $A$ for which $A^3 = 0$ but $A^2 \neq 0$. Hint: There are lots of them, and here is a method for constructing them. First, pick three linearly independent vectors in $R^3$, say $v_1, v_2, v_3$. Next, define a transformation $T : R^3 \rightarrow R^3$ as follows: $T(v_1) = v_2, \quad T(v_2) = v_3, \quad T(v_3) = 0$. If $A$ is the matrix of $T$ then $A^3 = 0$ but $A^2 \neq 0$. Can you prove this? Carry out the above procedure with $v_1 = (1, 1, 1), \quad v_2 = (0, 1, 1), \quad v_3 = (1, 0, -1)$. 