1. (20 points) Let \( T : P_3 \rightarrow M_{2 \times 2} \) be defined by

\[
T(at^3 + bt^2 + ct + d) = \begin{pmatrix} a + c & b + d \\ a - b + c - d & a + b + c + d \end{pmatrix}.
\]

(a) Find a basis for the kernel of \( T \).

**Solution:** To get the zero-matrix, we need \( a + c = 0, b + d = 0, a - b + c - d = 0, a + b + c + d = 0 \). We could form a \( 4 \times 4 \) matrix and reduce. However, the third and fourth equations are just combinations of the first and second, so all we really need are \( a + c = 0, b + d = 0 \). If we let \( a \) and \( b \) be our free variables, then \( c = -a, d = -b \). This means the polynomial \( at^3 + bt^2 + ct + d \) has to have the form \( at^3 + bt^2 - at - b = a(t^3 - t) + b(t^2 - 1) \). A basis for the kernel is \( \{t^3 - t, t^2 - 1\} \).

(b) Find a basis for the range of \( T \).

**Solution:** One way is to look for conditions on \( w, x, y, z \) so that there are \( a, b, c, d \) with \( \begin{pmatrix} a + c & b + d \\ a - b + c - d & a + b + c + d \end{pmatrix} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \). We need \( a + c = w, b + d = x, a - b + c - d = y, a + b + c + d = z \) to be consistent.

\[
\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

This means we need \( y = w - x \) and \( z = w + x \). That is, \( \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} w \\ x \\ w - x \\ w + x \end{pmatrix} \) = \( w \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + x \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \). A basis for the range is \( \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\} \).

A second approach some people used:

\[
\begin{pmatrix} a + c & b + d \\ a - b + c - d & a + b + c + d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + d \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.
\]

Noticing that the first matrix equals the third and the second equals the fourth, \( \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\} \) must be a spanning set for the range. Since these two matrices are independent, it is a basis.
2. (20 points) Two bases for $M_{2\times 2}$ are

\[ B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \quad \text{and} \quad C = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \]

(a) Find the transition matrix $P_{B \leftarrow C}$.

**Solution:** If the vectors in $B$ are $b_1, b_2, b_3, b_4$ and the vectors in $C$ are $c_1, c_2, c_3, c_4$, then we write the $c$-vectors as combinations of $b$-vectors, and the coefficients in this combinations give us the columns of $P$. We have $c_1 = b_4$, $c_2 = b_4 - b_1$, $c_3 = b_4 - b_2$ and $c_4 = b_4 - b_3$. Thus,

\[
P_{B \leftarrow C} = \begin{pmatrix} \begin{bmatrix} c_1 \end{bmatrix}_B & \begin{bmatrix} c_2 \end{bmatrix}_B & \begin{bmatrix} c_3 \end{bmatrix}_B & \begin{bmatrix} c_4 \end{bmatrix}_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \]

(b) Use part (a) to find $[A]_B$ given that $[A]_C = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

**Solution:**

\[
[A]_B = P_{B \leftarrow C} [A]_C = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}. \]

Here is a way to check: We use $[A]_B$ and $[A]_C$ to calculate $A$ and see if we get the same answer each time. First, $[A]_C$ tells us that $A = c_1 - c_2 + c_3 - c_4$ or

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \]

Using the $B$-basis instead, $A = b_1 - b_2 + b_3$ or

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \]
3. (20 points) Let \( A = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \).

(a) Find the characteristic polynomial of \( A \). Explain any fancy reasoning.

**Solution:** We have

\[
\det(xI - A) = \begin{vmatrix} x-2 & -2 & -2 & -2 \\ -2 & x-2 & -2 & -2 \\ -2 & -2 & x-2 & -2 \\ -2 & -2 & -2 & x-2 \end{vmatrix} = x^3 \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -2 & -2 & -2 & x-2 \end{vmatrix} = x^3(x-8).
\]

A second approach used by some people: In \( \det(xI - A) \), add the bottom three rows to the top. You get

\[
\det(xI - A) = \begin{vmatrix} x-2 & -2 & -2 & -2 \\ -2 & x-2 & -2 & -2 \\ -2 & -2 & x-2 & -2 \\ -2 & -2 & -2 & x-2 \end{vmatrix} = (x-8) \begin{vmatrix} 1 & 1 & 1 & 1 \\ -2 & x-2 & -2 & -2 \\ -2 & -2 & x-2 & -2 \\ -2 & -2 & -2 & x-2 \end{vmatrix} = (x-8) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix} = x^3(x-8).
\]

If we wanted to use theory to do this, the solution might go like this: We want a quartic. Since rows drop out when you reduce, we know 0 is an eigenvalue. Since the nullity of \( A \) is 3, there must be a factor of \((x-0)^3\) so there can only be one other factor, \( \det(xI - A) = x^3(x-c) \) for some \( c \). The sum of the eigenvalues is the trace of the matrix, so \( 0 + 0 + 0 + c = 2 + 2 + 2 + 2 \) or \( c = 8 \) and again,

\( \det(xI - A) = x^3(x-8) \).

(b) For each eigenvalue of \( A \), find a basis for its eigenspace. (There are only two eigenvalues. You might be able to guess them if you have trouble with part (a)).

**Solution:** The point of the hint: If we know there are only two eigenvalues, we might be able to guess them: Since \( A \) has a nullity, 0 is an eigenvalue. If
we multiply by a vector of all 1’s, we get a vector of all 8’s so 8 must be an eigenvalue. For the eigenspaces,

\[
A - 0I = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

One pivot means 3 free variables. We get basis \( \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \).

For 8, subtract 8 from the diagonal, put the bottom row on top to row reduce:

\[
A - 8I = \begin{pmatrix} -6 & 2 & 2 & 2 \\ 2 & -6 & 2 & 2 \\ 2 & 2 & -6 & 2 \\ 2 & 2 & 2 & -6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & -3 \\ 0 & -8 & 0 & 8 \\ 0 & 0 & -8 & 8 \\ 0 & 8 & 8 & -16 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Now the nullity is 1, the basis is \( \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \).

4. (20 points) Find the matrix for each of the following linear transformations.

(a) \( T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ x - y + z \end{pmatrix} \)

**Solution:** By inspection, \( T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ x - y + z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \). The matrix of the transformation is \( \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \).

(b) \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) satisfying \( T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \) and \( T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

**Solution:** The formula is \( A = \begin{pmatrix} T \begin{pmatrix} 1 \\ 0 \end{pmatrix} & T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \).
(c) \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) satisfying \( T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \) and \( T \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Hint: One way is to first figure out what \( T \) does to each standard basis vector.

\[
\text{Solution: Using the hint, } \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ so } T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = T\begin{pmatrix} 2 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ so } T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}. \text{ The matrix of the transformation is } \begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix}.
\]

An alternative some people used: Let the matrix be \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). We need
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \text{ This means } a + b = 2 \text{ and } 2a + b = 1 \text{ giving } a = -1, b = 3. \text{ Similarly, } c = -1, d = 3, \text{ giving the same answer.}
\]

5. (15 points) Let \( T : P_2 \to P_2 \) be a linear transformation for which
\[
T(1) = t^2 + t + 1, \quad T(t+1) = t^2 + 2t + 1, \quad T(t^2 + t + 1) = t^2 + 3t + 1.
\]

(a) Find a formula for \( T(at^2 + bt + c) \). Hint: One way is to first find \( T(1), T(t), T(t^2) \) and use that to find \( T(at^2 + bt + c) \).

\[
\text{Solution: We already know } T(1). \text{ To get } T(t) \text{ we use } t = (t + 1) - 1 \text{ so } T(t) = T(t+1) - T(1) = (t^2+2t+1) - (t^2+t+1) = t. \text{ Next, } t^2 = (t^2+t+1) - (t+1) \text{ so } T(t^2) = T(t^2 + t + 1) - T(t + 1) = (t^2 + 3t + 1) - (t^2 + 2t + 1) = t. \text{ Finally,}
\]
\[
T(at^2 + bt + c) = aT(t^2) + bT(t) + cT(1) = at + bt + c(t^2 + t + 1) = ct^2 + (a + b + c)t + c.
\]

(b) Find a vector in \( \text{Ker}(T) \). Hint: one way might be to use the formula from part (a). However, this could also be done by using the fact that \( t^2 + t + 1, t^2 + 2t + 1, t^2 + 3t + 1 \) are linearly dependent.

\[
\text{Solution: First, I was careless in writing this problem down! I had meant to ask for a non-zero vector in the kernel. Anyone who said 0 is always in the kernel got full credit. To answer the question I intended, if we use the formula we just got then we need } ct^2 + (a + b + c)t + c = 0 \text{ so } c = 0, a + b = 0. \text{ If we let } a = 1, b = -1, c = 0 \text{ we get the polynomial } t^2 - t \text{ as a vector in the kernel.}
\]
The approach I mentioned above is the following: Since
\[(t^2 + t + 1) - 2(t^2 + 2t + 1) + (t^2 + 3t + 1) = 0,
\]
so \(T(1) - 2T(t+1) + T(t^2 + t + 1) = 0\) or \(T(t^2 - t) = 0\) so \(t^2 - t\) is in the kernel.

6. (5 points) If \(T : V \rightarrow W\) is a linear transformation and \(T(u), T(v), T(w)\) are linearly independent in \(W\), prove that \(u, v, w\) must be linearly independent in \(V\).

**Solution:** My gut reaction to such problems: Start with: Suppose \(au + bv + cw = 0\). We need to make use of the information we were given, that \(T(u), T(v), T(w)\) are linearly independent. The easiest way to do this is to apply \(T\) to our supposed dependence relation. We have: \(0 = T(0) = T(au + bv + cw) = aT(u) + bT(v) + cT(w)\). Since \(T(u), T(v), T(w)\) are independent, \(a, b, c\) are each forced to be 0, showing \(u, v, w\) are independent.

I approach all such problems this way: form a combination of the vectors you want to be independent, set the combination equal to 0, and only then look at the information you were given to decide how best to proceed.

Some extra credit you can think about if you have time.

7. (3 points) With regard to problem 6, find a linear transformation \(T : M_{2\times2} \rightarrow M_{2\times2}\) and vectors \(u, v, w\) in \(M_{2\times2}\) for which \(T(u), T(v), T(w)\) are linearly independent but \(T\left(\begin{array}{cc}1 & 1 \\ 1 & 1 \end{array}\right) = 0\).

8. (3 points) Give a formula for a linear transformation \(T : P_3 \rightarrow P_3\) with \(\text{Range} = \text{Span}\{1 + t, \; t + t^2\}\) and \(\text{Kernel} = \text{Span}\{t + t^2, \; t^2 + t^3\}\).

9. (4 points) Suppose you have a \(2n \times 2n\) block matrix \(M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}\) where \(A\) is a matrix of all 2’s and \(B\) is a matrix of all 1’s. For example, when \(n = 2\), \(M = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\).

Find the characteristic polynomial of \(M\), its eigenvalues, and a basis for each eigenspace.