I thought I would give some examples of extending independent sets to a basis. For a first example, suppose we wish to extend \( \{ t^3 + t + 1, t^4 + 2t^2 + t \} \) to a basis for \( P_4 \). The usual method on a problem like this: Extend the set to a spanning set by appending the vectors from a known spanning set, and then remove dependent vectors, making sure to keep the ones in the original set. In a case like this, it is simplest to append the vectors for the standard basis for \( P_4 \) to get the spanning set \( \{ t^3 + t + 1, t^4 + 2t^2 + t, 1, t, t^2, t^3, t^4 \} \). We may also use the fact that every basis for \( P_4 \) consists of five vectors, so any 5-vector spanning set must be a basis. Thus, we must find two dependence relations to remove vectors. The great thing about a standard basis, is that it is easy to take any vector and write it as a linear combination of standard basis vectors. We make use of that fact.

Let’s label the vectors in our spanning set \( v_1, v_2, v_3, v_4, v_5, v_6 \) and \( v_7 \). One dependence relation is \( v_1 = v_3 + v_4 + v_6 \). We want to keep \( v_1 \), but that’s ok. We can view any vector in a dependence relation as depending on the others. So we can pick any of \( v_3, v_4, v_6 \) to remove. That is, we can rewrite things, say, \( v_1 = v_1 - v_3 - v_6 \) or \( v_6 = v_1 - v_3 - v_4 \), etc. I usually just select the largest index, so I will remove \( v_6 \) from our set. Next, \( v_2 = v_4 + 2v_5 + v_7 \), and so \( v_7 \) can also be removed. Our basis is \( \{ v_1, v_2, v_3, v_4, v_5 \} = \{ t^3 + t + 1, t^4 + 2t^2 + t, 1, t, t^2 \} \).

It is worth noting here that which vector you remove can make a difference. For example, suppose we remove \( v_4 \) at the first stage. Then \( v_4 \) can’t be used in a linear dependence relation at the next stage. But \( \{ v_1, v_2, v_3, v_5, v_6, v_7 \} = \{ t^3 + t + 1, t^4 + 2t^2 + t, 1, t^2, t^3, t^4 \} \) must be dependent. I found the following dependence by inspection (looking at it, an answer came to me). I got \( v_2 - v_1 = -v_3 + 2v_5 - v_6 + v_7 \). That is, subtracting \( v_1 \) from \( v_2 \) got rid of the coefficient of \( t \) and we had all the other powers of \( t \) to work with. Thus, we can get rid of any of \( v_3, v_5, v_6, v_7 \). If we get rid of \( v_6 \) just to be different, we would have a basis \( \{ v_1, v_2, v_3, v_5, v_7 \} = \{ t^3 + t + 1, t^4 + 2t^2 + t, 1, t^2, t^4 \} \).

What if we can’t just spot dependence relations? For example, in a vector space without a standard basis there might not be any obvious ways to get dependence relations. So let’s do this problem without using obvious dependencies. In this case, we set up a generic linear combination of the seven vectors, set that equal to 0, and use that to get a system of equations to help us out. I dislike subscripts so I will use the beginning of the alphabet for my scalars:

\[
a(t^3 + t + 1) + b(t^4 + 2t^2 + t) + c \cdot 1 + dt + et^2 + ft^3 + gt^4 = 0. \tag{1}
\]

Rewrite this as a polynomial:

\[
(b + g)t^4 + (a + f)t^3 + (2b + e)t^2 + (a + b + d)t + (a + c) = 0,
\]

and to be 0, a polynomial must have all coefficients equal to 0. This gives us our system of equations:

\[
a + c = 0, \quad a + b + d = 0, \quad 2b + e = 0, \quad a + f = 0, \quad b + g = 0.
\]

We get the coefficient matrix and reduce
and determined variables, so suppose we say \( p = 27 \)
\[
\begin{pmatrix}
 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 0 
\end{pmatrix} \Rightarrow \begin{pmatrix}
 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 0 
\end{pmatrix} \Rightarrow \begin{pmatrix}
 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
 0 & 0 & 0 & 0 & 1 & 0 & -2 
\end{pmatrix}
\]

This tells us that \( a = -f, \ b = -g, \ c = f, \ d = f + g, \ e = 2g \). How does this give us dependence relations? Pick \( f = 1, \ g = 0 \) and we get \( a = -1, \ b = 0, \ c = 1, \ d = 1, \ e = 0, \ f = 1, \ g = 0 \). That is, our equation (1) above becomes

\[-1(t^3 + t + 1) + 1 + t + t^3 = 0,
\]
or \( v_1 = v_3 + v_4 + v_6 \), as before. Setting \( f = 0, \ g = 1 \) gives the other dependence relation.

As a second example, let \( V \) be the space of all polynomials in \( P_3 \) that satisfy \( p(2) = 0 \) and \( p(3) = 0 \). The problem: Find a basis for \( V \), and extend this to a basis for \( W \), the set of polynomials in \( P_3 \) that only satisfy the condition \( p(3) = 0 \).

Finding a basis for \( V \): if \( p(x) = ax^3 + bx^2 + cx + d \) then we need \( 8a + 4b + 2c + d = 0 \) and \( 27a + 9b + 3c + d = 0 \). It is easier to use variables with the smallest coefficients as determined variables, so suppose we say \( a \) and \( b \) are free. Subtracting the first equation from the second, \( 19a + 5b + c = 0 \), so \( c = -19a - 5b \) and \( d = -2c - 8a - 4b = 30a + 6b \). As an aside, this would happen if we used row reduction, but with columns \( d, c, b, a \):

\[
\begin{pmatrix}
 1 & 2 & 4 & 8 \\
 1 & 3 & 9 & 27 
\end{pmatrix} \Rightarrow \begin{pmatrix}
 1 & 2 & 4 & 8 \\
 0 & 1 & 5 & 19 
\end{pmatrix} \Rightarrow \begin{pmatrix}
 1 & 0 & -6 & -30 \\
 0 & 1 & 5 & 19 
\end{pmatrix}
\]

We have \( at^3 + bt^2 + ct + d = at^3 + bt^2 - (19a + 5b)t + (30a + 6b) = a(t^3 - 19t + 30) + b(t^2 - 5t + 6) \). This means that \( \{t^3 - 19t + 30, t^2 - 5t + 6\} \) is a basis for \( V \). Now we want to extend this to a basis for \( W \). It turns out that \( W \) is 3-dimensional. If we knew this, then any three independent vectors in \( W \) would form a basis for \( W \), and so the task is to find one extra vector in \( W \), independent of the first two. In fact, \( t - 3 \) is in \( W \) and it is independent of the other two, so \( \{t^3 - 19t + 30, t^2 - 5t + 6, t - 3\} \) is such a basis. I will let you check the independence of these vectors. If we don’t know \( W \) is 3-dimensional, we can try to find a basis for it. Setting \( 27a + 9b + 4c + d = 0 \) and writing \( d = -27z - 9b - 3c \) we have that any vector in \( W \) can be written \( p(t) = at^3 + bt^2 + ct - 27a - 9b - 3c = a(t^3 - 27) + b(t^2 - 9) + c(t - 3) \) so \( W \) is 3-dimensional, with basis \( \{t^3 - 27, t^2 - 9, t - 3\} \). It turns out that any one of these vectors can be appended to the basis for \( V \) to get a new basis for \( W \). If we did not want to use facts about dimension, we could still do this problem: Given our bases \( \{t^3 - 19t + 30, t^2 - 5t + 6\} \) for \( V \) and \( \{t^3 - 27, t^2 - 9, t - 3\} \) for \( W \), the set \( \{t^3 - 19t + 30, t^2 - 5t + 6, t^3 - 27, t^2 - 9, t - 3\} \)
is a spanning set for $W$ so we just remove dependent vectors, while making sure to keep the first two. If we call these vectors $v_1$ through $v_5$, I actually checked (via a system of equations) that $5v_1 - 19v_2 - 5v_3 + 19v_4 = 0$ and $v_1 - v_3 + 19v_5 = 0$ so $v_4$ and $v_5$ depend on $v_1, v_2, v_3$, so these three form a basis.

One final example. Let $V$ be the the set of all $2 \times 2$ matrices who’s entries add to 0.
You should be able to check that this is a vector space. It has many nice bases, but suppose we are told that \[ \left\{ \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ 3 & -1 \end{pmatrix} \right\} \] is a basis. Our task: to find a basis that contains the matrix \[ \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}. \] We could proceed as follows: form the set containing this and the other three matrices: \[ \left\{ \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ 3 & -1 \end{pmatrix} \right\}, \] and look for a dependence relation (we need to remove one vector because dim($V$) = 3). Maybe a dependence relation can be found by inspection, but if not, we can use a system of equation. First, set a generic combination of the four matrices to 0:
\[ a \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} + b \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & -3 \\ 3 & -1 \end{pmatrix} = 0, \]
or
\[ \begin{pmatrix} a + b + c + d & a - b - 2c - 3d \\ a + b + c + 3d & -3a - b - d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \] Now we have four linear equations, we can form a matrix and reduce:
\[ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -2 & -3 \\ 1 & 1 & 1 & 3 \\ -3 & -1 & 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -3 & -4 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 3 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] This says that we need $a = \frac{1}{2}c, b = -\frac{3}{2}c$ and $d = 0$. If we pick $c = 2$ then our dependence relation is
\[ \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} - 3 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + 2 \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} = 0, \]
so we can eliminate either the second or third matrix, but not the fourth. One basis is
\[ \left\{ \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ 3 & -1 \end{pmatrix} \right\}. \]