1. Let \( A = \begin{pmatrix} 3 & 2 & 2 & 1 & 1 \\ 4 & 3 & 3 & 2 & 2 \\ 5 & 4 & 4 & 3 & 3 \end{pmatrix} \).

(a) Find the \( LU \) decomposition of \( A \).

(b) Find a basis for the null space of \( A \).

(c) Find a “nice” basis for the column space of \( A \).

(d) Write the fourth column of \( A \) as a linear combination of the first two.

(e) Describe the set of \( b \) for which \( Ax = b \) is consistent.

2. Find the inverse of \( A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix} \) in two ways: by row reduction, and by the adjoint formula.

3. Find the determinant of the matrix \( A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 3 & 5 & 4 \\ 2 & 6 & 9 & 9 \\ 2 & 4 & 6 & 12 \end{pmatrix} \).

4. Let \( V = \{(w, x, y, z) \mid w + x = y + z\} \).

(a) Prove that \( V \) is a subspace of \( \mathbb{R}^4 \).

(b) Find a basis for \( V \).

(c) Find a basis for \( V \) which contains the vector \((1, 1, 1, 1)\).

5. Let \( T : P_2 \to P_2 \) be defined by \( T(p(t)) = p(t) + p'(t) \).

(a) Show that \( T \) is a linear transformation.

(b) Show that \( T \) is not diagonalizable

6. Two bases for \( P_2 \) are \( B = \{(1 + t)^2, 1 + t, 1\} \) and \( C = \{1, 1 - t, (1 - t)^2\} \). Find the transition matrices from \( B \) to \( C \) and \( C \) to \( B \). Make sure you know which is which!

7. Determine which of the following matrices is diagonalizable. Justify each answer.

\[(a) \quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad (b) \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[(c) \quad C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & -1 & -1 & -1 \end{pmatrix} \quad (d) \quad D = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & -1 & 1 & -1 \end{pmatrix} \]
8. Is there a linear transformation from $P_2$ to $P_2$ with the following properties? If so, give an example of such a transformation. Otherwise, prove there is no such transformation.
   (a) $T(t^2 + t + 1) = t^2 + t + 1$, $T(t^2 + 2t + 3) = 3t^2 + 2t + 1$, $T(t^2 + 2t + 2) = 2t^2 + 2t + 1$
   (b) $T(t^2 + t + 1) = t^2 + t + 1$, $T(t^2 + 2t + 3) = 3t^2 + 2t + 1$, $T(t^2 + 3t + 5) = 5t^2 + 3t + 1$
   (c) $T(t^2 + t + 1) = t^2 + t + 1$, $T(t^2 + 2t + 3) = 3t^2 + 2t + 1$, $T(t^2 + 3t + 5) = 2t^2 + 2t + 1$

9. For each linear transformation $T$, find bases for the kernel and the range. Then find a basis $B$ for which $[T]_B$ is diagonal.
   (a) $T : M_{2 \times 2} \rightarrow M_{2 \times 2}$ defined by $T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + d & b + c \\ b + c & a + d \end{pmatrix}$.
   (b) $T : P_2 \rightarrow P_2$ defined by $T(p(t)) = (t + 1)^2 p''(t) - (t + 1)p'(t) + 3p(t)$.

10. (a) Show that $T : P_{10} \rightarrow P_{10}$ defined by $T(p(t)) = p(t + 1)$ is NOT diagonalizable.
    (b) Show that $T : P_{10} \rightarrow P_{10}$ defined by $T(p(t)) = p(2t + 1)$ IS diagonalizable.

11. A linear transformation $T$ for which $T(T(v)) = v$ for all $v$ is called a reflection.
    (a) Show that $T : P_n \rightarrow P_n$ defined by $T(p(t)) = p(1 - t)$ is a reflection.
    (b) For the case of $n = 3$ in part (a), find all eigenvalues and a basis for each eigenspace of $T$.
    (c) In general, prove that if $T$ is a reflection, then its only possible eigenvalues are 1 and -1.

12. An $X$-Matrix is a matrix of the form $\begin{pmatrix} a & 0 & a \\ 0 & b & 0 \\ c & 0 & c \end{pmatrix}$.
    (a) Show that the set of all $X$-Matrices is a subspace of $M_{3 \times 3}$.
    (b) Find a basis for the set of all $X$-Matrices.
    (c) Give an example of an $X$-Matrix which is diagonalizable.
    (d) Give an example of an $X$-Matrix which is not diagonalizable.

13. Consider the transformation $T : M_{3 \times 2} \rightarrow M_{3 \times 2}$ that takes a $M_{3 \times 2}$ matrix and turns it upside down. for example, $T \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{pmatrix}$.
    (a) Show that $T$ is a reflection (see problem 11).
    (b) Find a bases for each eigenspace of $T$.

14. Let $T : V \rightarrow W$ be a linear transformation.
    (a) If $\{u, v\}$ is linearly independent in $V$, does $\{T(u), T(v)\}$ have to be linearly independent in $W$? If so, prove it, if not, give a counter example.
    (b) If $\{T(u), T(v)\}$ is linearly independent in $W$, does $\{u, v\}$ have to be linearly independent in $V$? If so, prove it, if not, give a counter example.