Math 2326
Fall 2008

Logical Equivalence

Consider the truth tables for $p \rightarrow q$ and $\neg p \vee q$:

p	q	$p\rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We see that the $p \rightarrow q$ and $\neg p \vee q$ columns are the same. It would seem funny to say that $p \rightarrow q = \neg p \vee q$, so we invent a new term. Two propositions are said to be **logically equivalent** if they have the same truth value for all possible truth values of the constituents. (Said differently, they are logically equivalent if their Truth Tables are the same.) We will **not** write $p \rightarrow q = \neg p \vee q$, but we will write $p \rightarrow q \equiv \neg p \vee q$, with "$\equiv$" having the meaning "is logically equivalent to."

Related ideas are the idea of a tautology, and a contradiction. A **tautology** is a proposition that is always true (it's truth table has nothing but T in its column). A **contradiction** is always false (nothing but F in its column). The simplest example of a tautology is $p \vee \neg p$. For a contradiction, $p \wedge \neg p$. Technically, two compound propositions, P and Q are called logically equivalent if $P \leftrightarrow Q$ is a tautology. Below is a list of important logical equivalences:

Name	Law		Name	Law
Domination	$p \vee T \equiv T$		Idempotence	$p \vee p \equiv p$
	$p \wedge F \equiv F$			$p \wedge p \equiv p$
Identity	$p \vee F \equiv p$		Commutivity	$p \vee q \equiv q \vee p$
	$p \wedge T \equiv p$			$p \wedge q \equiv q \wedge p$
Double Negation	$\neg(\neg p) \equiv p$			

Name	Law
Associativity	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
Distributive Laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan's Laws	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
	$\neg(p \wedge q) \equiv \neg p \vee \neg q$

In addition to the above, there are several important equivalences involving the implication and the biconditional. These are:

$$p \to q \equiv \neg p \vee q, \quad \neg(p \to q) \equiv p \wedge \neg q, \quad p \to q \equiv \neg q \to \neg p$$

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p), \quad p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q).$$

With the use of logical equivalences, you can show things without using a truth table. For example, a logical rule called the **Law of Simplification** says $(p \wedge q) \to p$ is a tautology. Here are two proofs, the first using truth tables:

p	q	p∧q	(p ∧ q) → p
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Alternatively, using logical equivalences, we have

$$(p \wedge q) \to p \equiv \neg(p \vee q) \vee p \equiv (\neg p \vee \neg q) \vee p$$
$$\equiv (\neg q \vee \neg p) \vee p \equiv \neg q \vee (\neg p \vee p) \equiv \neg q \vee T \equiv T.$$

Here, $P \equiv T$ means that P is a tautology.