Math 2326

Fall 2008

Nested Quantifiers

A note on proving things: To prove $\exists$ x P(x), one need only give an example of an x for which P(x) is true. Similarly, to demonstrate that $\forall$ x P(x) is false, an example of an x making P(x) false is all that is needed. Such an x is called a **counter example**. Proving $\exists$ x P(x) is false or $\forall$ x P(x) is true is much harder. We will come back to these later.

In mathematics, we usually have lots of variables floting around. For example, from Section 1.4, we learned that A**x** = **b** is always consistent if A has a pivot in every row. Here we seem to be swimming in variables: Matrices A, vectors **b**, the rows of A, and a hidden one: A**x** = **b** is consistent means that there is a vector $\mathbf{x_0}$ with the property that when you multiply it by A, you get **b**. All this without even mentioning the constituents of **x**, that we might typically refer to as variables. How do we handle this kind of thing? We must quantify every variable that appears in a propositional function.

To back up to something simpler, consider the statement

$$(x + 1)^2 = x^2 + 2x + 1.$$

What we mean by this is that $\forall x\ ((x + 1)^2 = x^2 + 2x + 1)$. That is, the original statement is not a propostion, but a propositional function. To make it a proposition, we quantify the variable x. Extending this,

$$(x + y)^2 = x^2 + 2xy + y^2$$

means

$$\forall x\ \forall y\ ((x + y)^2 = x^2 + 2xy + y^2).$$

We quantify each of the two variables. We might read the above as “For every x and every y, ...”

Often, a mathematical statement does not sound like a multivariable quantification, but the only reasonable way to think about it is in those terms. For example, if someone tells you that a positive number times a negative number is negative, what they almost certainly mean is the following:

$$\forall x \forall y ((x > 0 \wedge y < 0) \rightarrow xy < 0).$$

That is, a universal claim was being made. In such cases, we say the quantifiers are **nested**. Here is another: If a prime is one more than a multiple of 4, then it is a sum of two squares. We might write this:

$$\forall p (\exists k (p = 4k + 1) \rightarrow \exists m \exists n (p = m^2 + n^2)).$$

Here, we quantify over all primes p. The first statement inside the $\forall p(\ )$ deals with the property that p is one more than a multiple of 4. The conclusion is that there are integers floting around with p being the sum of their squares.

Closer to this course, what does it mean to say that you can row reduce a matrix? It means for every matrix A there is a matrix R with the property that R is in reduced echelon form and A is row equivent to R. (Slightly) more symbolic, $\forall A \exists R$ (R is reduced $\wedge$ A is row equivalent to R).

When you have several variables, order matters. That is, consider the following: (a) $\forall x \exists y (x + y > 0)$ and (b) $\exists y \forall x (x + y > 0)$. These are not the same. In fact, (a) is true and (b) is false. Intuitively, you might think of working from the outside in. That is, in (a), x comes first, and then y gets its chance. That is, when picking y, you can assume x is known. However in (b), y comes first. Given some value of y, is $x + y > 0$ true for all x?

Technically, the way something like $\forall x \exists y (x + y > 0)$ works is that we view $\exists y (x + y > 0)$ as a propositional function in x, the variable with no quantifier yet. That is to say, let P(x) be the propositional function $\exists y (x + y > 0)$. Then P(1) is the proposition $\exists y (1 + y > 0)$, P(-10) is the propositional function $\exists y (-10 + y > 0)$, and so on. We see that these are true, and in general, $\forall x\, P(x)$ is true.

With $\exists y \forall x (x + y > 0)$, we first consider the propositional functionQ(y): $\forall x (x + y > 0)$. Here, Q(1) says $\forall x (x + 1 > 0)$, Q(-4) says $\forall x (x - 4 > 0)$. These are both false. In particular, Q(y) is false for all y, so $\exists y\, Q(y)$ is false.

How do we negate nested quantifiers? In a fashion similar to ordinary quantifiers: the negation changes $\forall$ to $\exists$, $\exists$ to $\forall$ and then moves inside one level. For example,

$$\begin{aligned} \neg(\forall x \exists y (x + y > 0)) &\equiv \exists x \neg(\exists y (x + y > 0)) \\ &\equiv \exists x \forall y \neg(x + y > 0) \\ &\equiv \exists x \forall y (x + y \leq 0). \end{aligned}$$

For a more complicated example,

$$\neg(\forall p (\exists k (p = 4k + 1) \to \exists m \exists n (p = m^2 + n^2)))$$

$$\equiv \exists p \neg((\exists k (p = 4k + 1) \to \exists m \exists n (p = m^2 + n^2)))$$

$$\equiv \exists p ((\exists k (p = 4k + 1) \wedge \neg(\exists m \exists n (p = m^2 + n^2))))$$

$$\equiv \exists p ((\exists k (p = 4k + 1) \wedge \forall m \neg(\exists n (p = m^2 + n^2))))$$

$$\equiv \exists p ((\exists k (p = 4k + 1) \wedge \forall m \forall n \neg(p = m^2 + n^2)))$$

$$\equiv \exists p ((\exists k (p = 4k + 1) \wedge \forall m \forall n (p \neq m^2 + n^2))).$$

Here, the meaning is that the original claim about primes is wrong if there is a prime which IS one more than a multiple of 4, but which is never the sum of two squares.