1. Determine, with reasons, which of the following matrices is diagonalizable.

(a) $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$  
(b) $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$  
(c) $C = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix}$  
(d) $D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$  
(e) $E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & -1 & -1 & -1 \end{pmatrix}$  
(f) $F = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & -1 & 1 & -1 \end{pmatrix}$

2. A linear operator $T$ satisfying $T^k = 0$ for some $k$ is called a nilpotent operator.

(a) Prove that the only nilpotent operator that is diagonalizable is the zero operator.

(b) Give an example of a nonzero nilpotent operator on $P^3$.

(c) Suppose that $V$ is $n$-dimensional and that $T^n = 0$ but $T^{n-1} \neq 0$. If $v$ is any vector with the property that $t^{n-1}(v) \neq 0$ show that $B = \{v, T(v), T^2(v), \ldots, T^{n-1}(v)\}$ is a basis for $V$.

(d) Find the matrix of $T$ with respect to the basis $B$ in part (c).

3. Can you give an example of a $10 \times 10$ matrix $A$ with $A^6 = 0$ but $A^5 \neq 0$?

4. Let $T : F^{2 \times 2} \rightarrow F^{2 \times 2}$ be defined by $T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - b + c + d & -a + b + c + d \\ c - d & d - c \end{pmatrix}$.

(a) Find the matrix of $T$ with respect to the standard basis.

(b) If $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, it turns out that $B = \{v, T(v), T^2(v), w\}$ is a basis for $F^{2 \times 2}$. Find the matrix of $T$ with respect to $B$.

(c) Find the characteristic polynomial of $T$, all eigenvalues, and a basis for each eigenspace.

(d) Is $T$ diagonalizable? Explain.

5. Let $\langle u, v \rangle$ be the Euclidean inner product on $R^3$.

(a) If $U$ is the subspace spanned by $(1, 1, 0)$, find a basis for $U^\perp$.

(b) Find the matrix for the orthogonal $R^3$ onto $U$.

(c) Find the matrix for the orthogonal projection of $R^3$ onto $U^\perp$.

(d) If $V$ is the subspace spanned by $(1, 1, 1), (1, 2, 3), (1, 3, 5)$, find a basis for $V^\perp$.

(e) Find an orthogonal basis for $V$. 
6. Using the euclidean inner product on $\mathbb{R}^3$, find an orthogonal basis containing the vector $(1, 1, 1)$.

7. Find orthogonal bases for $P^2$ with respect to the following inner products.
   (a) $\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx$
   (b) $\langle p(x), q(x) \rangle = \int_{0}^{1} p(x)q(x)dx$

8. Here is a nice way to create inner product spaces from other inner product spaces. Let $V$ be an inner product space with inner product $\langle u, v \rangle$.
   (a) If $T$ is an invertible linear operator on $V$, prove that $\langle u, v \rangle_T \overset{\text{def}}{=} \langle T(u), T(v) \rangle$ is also an inner product.
   (b) What goes wrong if $T$ is not invertible?
   (c) Find an inner product on $\mathbb{R}^2$ with $\langle (1, 2), (2, 1) \rangle = 0$.

9. Let $V$ be an inner product space.
   (a) If $u$ and $v$ are nonzero vectors in $V$, prove that $\operatorname{proj}_u(x) + \operatorname{proj}_v(x)$ is a projection if and only if $u$ is orthogonal to $v$.
   (b) If $U$ and $W$ are subspaces of $V$ when is $\operatorname{proj}_U(x) + \operatorname{proj}_W(x)$ a projection?

10. If $V$ is a 4-dimensional inner product space and $u$ and $v$ are orthogonal, what can you say about the eigenvalues, eigenvectors, and characteristic polynomial of $T(x) = 2 \operatorname{proj}_u(x) + 4 \operatorname{proj}_v(x)$? Is $T$ diagonalizable?