1. Let \( A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -3 \\ -4 & 3 & 6 \end{pmatrix} \).

   (a) Find the determinant of \( A \) by a cofactor expansion down the second column.

   (b) Use the adjoint formula to find \( A^{-1} \).

2. Consider the sequenced of matrices

   \[
   M_2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 7 \end{pmatrix}, \ldots
   \]

   Find a formula for \( \det(M_n) \) as a function of \( n \).

3. Consider the block matrix \( M = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix} \), where \( A \) and \( B \) are square matrices.

   (a) Show that \( D(A) = \det(M) \), where \( B \) and \( C \) are fixed is \( n \)-linear and alternating in the rows of \( A \).

   (b) Use part (a) to find a formula for \( \det(M) \). Note: if \( A \) is \( m \times m \) and \( B \) is \( n \times n \), the result might depend on \( m \) and \( n \).

4. Determine, with reasons, which of the following matrices is diagonalizable.

   (a) \( A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \)
   
   (b) \( B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \)

   (c) \( C = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \)

   (d) \( D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \)

   (e) \( E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \\ 3 & -1 & -1 \end{pmatrix} \)

   (f) \( F = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & -1 & 1 & -1 \end{pmatrix} \)

5. A linear operator \( T \) satisfying \( T^k = 0 \) for some \( k \) is called a \textbf{nilpotent} operator.

   (a) Prove that the only nilpotent operator that is diagonalizable is the zero operator.

   (b) Give an example of a nonzero nilpotent operator on \( P^3 \).

   (c) Suppose that \( V \) is \( n \)-dimensional and that \( T^n = 0 \) but \( T^{n-1} \neq 0 \). If \( v \) is any vector with the property that \( t^{n-1}(v) \neq 0 \) show that \( B = \{ v, T(v), T^2(v), \ldots, T^{n-1}(v) \} \) is a basis for \( V \).

   (d) Find the matrix of \( T \) with respect to the basis \( B \) in part (c).
6. Can you give an example of an $8 \times 8$ matrix $A$ with $A^6 = 0$ but $A^5 \neq 0$?

7. Let $T : F^{2 \times 2} \to F^{2 \times 2}$ be defined by $T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - b + c + d & -a + b + c + d \\ c - d & d - c \end{pmatrix}$.

   (a) Find the matrix of $T$ with respect to the standard basis.

   (b) If $v = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, it turns out that $B = \{v, T(v), T^2(v), w\}$ is a basis for $F^{2 \times 2}$. Find the matrix of $T$ with respect to $B$.

   (c) Find the characteristic polynomial of $T$, all eigenvalues, and a basis for each eigenspace.

   (d) Is $T$ diagonalizable? Explain.

8. Give an example of a block upper triangular matrix $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ for which the minimal polynomial of $M$ is not the least common multiple of the minimal polynomials for $A$ and $C$.

9. Prove that any matrix which commutes with $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is diagonalizable. Is the same true for any matrix which commutes with $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$?