From the book: 4.5.4, 4.5.8, 4.5.12 (Sections 4.5). Also do the following:

1. Find an orthogonal matrix which diagonalizes
\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\].

2. Let \( \langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x) \, dx \). Define \( T \) on \( P^2 \) by
\[
T(ax^2 + bx + c) = \frac{2a + 5c}{7} x^2 + bx + \frac{2a + 5c}{7}.
\]
(a) Show that \( T \) is a Hermitian operator.
(b) Find an orthogonal basis of eigenvectors for \( T \).

3. Care is needed not to jump to conclusions with Hermitian operators, even on a space as simple as \( \mathbb{R}^2 \). Consider the inner product \( \langle (w, x), (y, z) \rangle = (w - x)(y - z) + xz \) on \( \mathbb{R}^2 \).
(a) Use Gram-Schmidt to convert \{ (1, 0), (0, 1) \} into an orthonormal basis for \( \mathbb{R}^2 \) with respect to this inner product.
(b) Define \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( T(x, y) = (11x - 3y, 2x + 4y) \). Show that \( T \) is a Hermitian operator with respect to this inner product, even though the standard matrix of \( T \) is not symmetric.
(c) Find the matrix of \( T \) with respect to the orthonormal basis of part (a).
(d) Find an orthonormal basis of eigenvectors for the transformation in part (b), using this inner product.
(e) Find the transition matrices between the bases you found in parts (a) and (c).

For extra credit:

4. Prove that every orthogonal projection is a Hermitian operator. That is, if \( V \) is an inner product space with subspace \( W \), prove that \( T(v) = \text{proj}_W(v) \) is a Hermitian operator.