1. Suppose that \( \{u, v, w\} \) is a linearly independent set in some vector space \( V \).

   (a) Show that \( \{u + v, u + v + w, u + v + 2w\} \) is linearly dependent.

   **Solution:** A dependence relation is \((-1)(u + v) + 2(u + v + w) = u + v + 2w\).

   (b) Find a basis for \( \text{Span}\{u + v, u + v + w, u + v + 2w\} \).

   **Solution:** One basis is \( \{u + v, u + v + w\} \), found by deleting a dependent vector. As mentioned in class, deleting a dependent vector from a set does not change the set’s span. We still need independence. For this, suppose that \( x(u + v) + y(u + v + w) = 0 \), which we rewrite \((x + y)u + (x + y)v + yw = 0\). Since \( \{u, v, w\} \) is independent it must be that the coefficients here, \( x + y \), and \( y \) are both 0. But then \( y = 0 \rightarrow x = 0 \), showing only the trivial combination sums to 0.

   (c) Carefully show that \( \{u + v, u + v + w, u - v + w\} \) is linearly independent.

   **Solution:** Similar to the above: if \( x(u + v) + y(u + v + w) + z(u - v + w) = 0 \) then \((x + y + z)u + (x + y - z)v + (y + z)w = 0\). The independence of \( \{u, v, w\} \) gives us a system of equations \( x + y + z = 0 \), \( x + y - z = 0 \), \( y + z = 0 \). Subtracting the third equation from the first, \( z = 0 \rightarrow y = 0 \rightarrow x = 0 \). Since the coefficients are 0 by force, the given vectors are independent.

   (d) Prove that \( \text{Span}\{u, v, w\} = \text{Span}\{u + v, u + v + w, u - v + w\} \).

   **Solution:** Let \( V = \text{Span}\{u, v, w\} \), \( W = \text{Span}\{u + v, u + v + w, u - v + w\} \). The goal is to show that \( V = W \). Since \( x(u + v) + y(u + v + w) + z(u - v + w) = (x + y + z)u + (x + y - z)v + (y + z)w \), every linear combination of the \( u + v, u + v + w, u - v + w \) is also a combination of \( u, v, w \). This means \( W \subseteq V \).

   The quickest approach is to now say \( W \) is a 3-dimensional subspace (by part (c)) of \( V \), itself 3-dimensional, so \( W = V \).

   Alternatively, we show that \( V \subseteq W \) by showing that every linear combination of \( u, v, w \) is also a combination of \( u + v, u + v + w, u - v + w \). To that end, given \( xu + yv + zw \) we want \( a, b, c \) so that \( xu + yv + zw = a(u + v) + b(u + v + w) + c(u - v + w) \). We rewrite the right hand side \((a + b + c)u + (a + b - c)v + (b + c)w \), and by independence, we have \( a + b + c = x \), \( a + b - c = y \), \( b + c = z \). We need to show a solution exists in \( a, b, c \). We have \( a = x - z, c = \frac{1}{2}x - \frac{1}{2}y, b = z - c = -\frac{1}{2}x + \frac{1}{2}y + z \). Thus, we can solve for \( a, b, c \), and \( V \subseteq W \), giving \( V = W \).
2. Find bases for each of the following vector spaces.

(a) $U =$ the set of all points $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$ with $w + x = y + z$.

Solution: We need $w = -x + y + z$, giving

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$  

(b) $V_1 =$ the set of all $3 \times 3$ matrices $A$ with $A = -A^t$.

Solution: $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} -a & -d & -g \\ b & -e & -h \\ -c & -f & -i \end{pmatrix}$. This means the diagonal entries must be 0 and the entries below the diagonal are the negatives of those above the diagonal. $A = \begin{pmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{pmatrix} = b \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + f \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

A basis is

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

(c) $V_2 =$ the set of all $4 \times 2$ matrices $A$ with $A \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = 0$.

Solution: $\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0$ when $a = b, c = d, e = f, g = h$. This means the set is the set of all matrices of the form $\begin{pmatrix} a & a \\ c & d \\ e & e \\ g & g \end{pmatrix}$, with basis

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$
(d) $W = \{ p(x) \in P^4 \mid p(1) = p(-1) \}$.

**Solution:** If $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ then we need $a + b + c + d + e = a - b + c - d + e$, or $b + d = 0$. That is, the polynomial has the form $ax^4 + bx^3 + cx^2 - bx + e$, and a basis for the space is $\{ x^4, x^3 - x, x^2, 1 \}$.

3. $B = \{ 1, x - 1, (x - 1)^2, (x - 1)^3 \}$ and $C = \{ x^3, x^3 + x^2, x^2 + x, x + 1 \}$ are bases for $P^3$.

(a) Let $p(x) = x^3 - 2x^2 + 3x - 4$. find the coordinate vectors $[p(x)]_B$ and $[p(x)]_C$.

**Solution:** The defining relation is $[p(x)]_B = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ when $p(x) = a + b(x - 1) + c(x - 1)^2 + d(x - 1)^3$. Either solving the equations or using a Taylor expansion, $a = -2, b = 2, c = 1, d = 1$ so $[p(x)]_B = \begin{pmatrix} -2 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. For $[p(x)]_C$ we need $a, b, c, d$ with $x^3 - 2x^2 + 3x - 4 = ax^3 + b(x^3 + x^2) + c(x^2 + x) + d(x + 1)$. This gives the system $a + b = 1, b + c = -2, c + d = 3, d = -4$, with solution $a = 10, b = -9, c = 7, d = -4$ so $[p(x)]_C = \begin{pmatrix} 10 \\ -9 \\ 7 \\ -4 \end{pmatrix}$.

(b) Suppose that $q(x)$ is a polynomial with $[q(x)]_C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Find $[q(x)]_B$.

**Solution:** Simplest is to first find $q(x): q(x) = x^3 + (x^3 + x^2) + (x^2 + x) + (x + 1) = 2x^3 + 2x^2 + 2x + 1$. Then we find the $B$-coordinates of this giving $[q(x)]_B = \begin{pmatrix} 7 \\ 12 \\ 8 \\ 2 \end{pmatrix}$.

(c) Find the transition matrices $P_{B \leftarrow C}$ and $P_{C \leftarrow B}$

**Solution:** The formula is $P_{C \leftarrow B} = ([b_1]_C | [b_2]_C | [b_3]_C | [b_4]_C)$, where the $b_i$ are the $B$-basis vectors. That is, to get $P_{C \leftarrow B}$, we get the $C$-coordinates of each of $B$’s vectors and put them in as columns of a matrix. In this case, we get the matrix
\[
P_{C\leftarrow B} = \begin{pmatrix}
-1 & 2 & -4 & 8 \\
1 & -2 & 4 & -7 \\
-1 & 2 & -3 & 4 \\
1 & -1 & 1 & -1
\end{pmatrix}.
\]
To get \( P_{B\leftarrow C} \) we can either invert \( P_{C\leftarrow B} \) or use the formula
\[
P_{B\leftarrow C} = \begin{pmatrix}
[c_1]_B | [c_2]_B | [c_3]_B | [c_4]_B
\end{pmatrix}.
\]
I tend to prefer this second approach.

We have \( P_{B\leftarrow C} = \begin{pmatrix}
1 & 2 & 2 & 2 \\
3 & 5 & 3 & 1 \\
3 & 4 & 1 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix} \). Multiplying the two matrices together does give the identity, which gives a check on our calculations.

4. Let \( V = \mathbb{R}^{2\times 4} \), the set of all \( 2 \times 4 \) matrices. Let \( U \) be the subspace of all matrices of the form \( \begin{pmatrix}
a & b & b & a \\
c & d & d & c
\end{pmatrix} \) and \( V \) the subspace of all matrices of the form \( \begin{pmatrix}
a & b & c & d \\
a & b & c & d
\end{pmatrix} \). Find bases for \( U \cap V \) and \( U + V \).

**Solution:** For \( U \cap V \), to be in both \( U \) and \( V \) we need a matrix \( A \) to have the form
\[
\begin{pmatrix}
a & b & b & a \\
c & d & d & c
\end{pmatrix} = a \begin{pmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1
\end{pmatrix} + b \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix},
\]
so \( U \cap V \) is 2-dimensional, with basis \( \begin{Bmatrix}
\begin{pmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1
\end{pmatrix}, \\
\begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix}
\end{Bmatrix} \).

We proved in class that \( \dim(U + V) + \dim(U \cap V) = \dim(U) + \dim(V) \). This means \( \dim(U + V) = 4 + 4 - 2 = 6 \). I did not mention this in class but one way to find a basis for \( U + V \) is to take the union of bases for \( U \) and \( V \), and then remove dependent vectors. That is, the union of bases will give a spanning set for \( U + V \), and every spanning set has a basis as a subset. In our case, a spanning set would consist of
\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}, \\
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \\
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \\
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
If we call these vectors \( v_1 \) through \( v_8 \) then we must remove two dependent vectors to get 6 for a basis. Since \( v_1 + v_3 = v_5 + v_8 \) and \( v_2 + v_4 = v_6 + v_7 \), we can remove the last two matrices and keep the first 6 for a basis.

An alternative would be to follow the proof of the theorem. If the basis vectors for \( U \cap V \) are \( u_1 \) and \( u_2 \), then a basis for \( U \) is \( \{u_1, u_2, v_1, v_2\} \), and a basis for \( V \) is \( \{u_1, u_2, v_5, v_6\} \). A basis for \( U + V \) is now the union of these, \( \{u_1, u_2, v_1, v_2, v_5, v_6\} \).