Problem 3.1.4 Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations. Find the standard matrix for $g \circ f$ if

(a) $g(x, y) = (x - y, 0)$ and $f(x, y, z) = (z - x, z - y)$.

**Solution:** There are three approaches on a problem like this: We could proceed directly. For example, since $g \circ f$ is a transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$, we need to know what it does to the three standard basis vectors. We have $(g \circ f)(e_1) = (g \circ f)(1, 0, 0) = (z - x, z - y) = (1, 0)$ and $(g \circ f)(e_2) = (g \circ f)(e_3) = (0, 0)$ so the matrix of the transformation is \[
\begin{pmatrix}
-1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Or, we could get a formula for $g \circ f$ first: $(g \circ f)(x, y, z) = g(f(x, y, z)) = g(z - x, z - y) = (y - x, 0)$, and use this to get the matrix by substituting in $e_1, e_2, e_3$.

Finally, the matrix of $g \circ f$ is the product of the matrices for $g$ and $f$ giving

$$[g \circ f] = [g][f] = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) $f(x, y, z) = (0, 0)$

**Solution:** Since $g \circ f$ is the 0-function here, the matrix is \[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

The only important thing is to get the dimensions right.

(c) $f(x, y, z) = (x, x)$ and the matrix of $g$ is \[
\begin{pmatrix}
2 & 3 \\
2 & 3
\end{pmatrix}.
\]

**Solution:** Easiest is to multiply the matrices for the transformations:

$$[g \circ f] = [g][f] = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix}.$$
Problem 3.1.6 Suppose that $T : V \to R^n$ and $U : V \to R^m$ are linear. Show that the function $G : V \to R^{n+m}$ defined by $G(v) = (T(v), U(v))$ is a linear transformation.

Solution: We show that $G(u + cv) = G(u) + cG(v)$. We have

$$G(u + cv) = (T(u + cv), U(u + cv)) = (T(u) + cT(v), U(u) + cU(v))$$
$$= (T(u), U(u)) + c(T(v), U(v))$$
$$= G(u) + cG(v).$$

Problem 3.2.14 Suppose that $R, S : V \to W$ are both linear transformations. Define

$$(R + S)(v) = R(v) + S(v).$$

Show that:

(a) $R + S : V \to W$ is a linear transformation.

Solution: This is similar to the problem above:

$$(R + S)(u + cv) = R(u + cv) + S(u + cv) = R(u) + cR(v) + S(u) + cS(v)$$
$$= R(u) + S(u) + c(R(v) + S(v))$$
$$= (R + S)(u) + c(R + S)(v).$$

(b) $\text{Rank}(R) + \text{Rank}(S) \geq \text{Rank}(R + S)$.

Solution: Suppose that the range of $R$ is spanned by independent vectors $w_1, \ldots, w_k$ and the range of $S$ is spanned by independent vectors $w_{k+1}, \ldots, w_{k+m}$. This means the rank of $R$ is $k$ and the rank of $S$ is $m$. Note that the first set of vectors can have elements in common with the second set. In any case, the range of $R + S$ is a subspace of Span\{ $w_1, \ldots, w_k, w_{k+1}, \ldots, w_{k+m}$ \}, so its dimension is at most $k+m$. To show the given set is a spanning set, if $w$ is in the range of $R + S$ then $w = (R + S)(v)$ for some $v \in V$, and $(R + S)(v) = R(v) + S(v)$. Now $R(v)$ is a combination of the first set of $w$’s, and $S(v)$ is a combination of the second set, so $R(v) + S(v)$ is a combination of their union.
Also do the following:

1. Recall that if \( v \) is a column vector of the right size and \( A \) is an \( m \times n \) matrix, then \( A_k(v) \) is the matrix obtained by replacing the \( k \)’th column of \( A \) by \( v \). Define \( T : F^3 \rightarrow F^3 \) as follows. Let \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \). Then \( T(v) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \), where \( a = \text{det}(A_1(v)), \ b = \text{det}(A_2(v)), \ c = \text{det}(A_3(v)) \).

(a) Find a basis for the kernel of \( T \).

**Solution:** For the kernel, we need \( \text{det}(A_1(v)) = \text{det}(A_2(v)) = \text{det}(A_3(v)) = 0 \). Now \( \text{det}(A_3(v)) = 0 \) if and only if \( v \) is a combination of the first two columns of \( A \). That is, if we view \( \text{det}(A_3(v)) \) as a transformation from \( F^3 \) to \( F \), then the kernel has the first two columns of \( A \) as a basis. If these columns are \( v_1 \) and \( v_2 \), then a simple check shows that \( \text{det}(A_1(v_1)) = \text{det}(A_1(v_2)) = 0 \) and \( \text{det}(A_2(v_1)) = \text{det}(A_2(v_2)) = 0 \). This means the that \( v_1 \) and \( v_2 \) form a basis for the kernel of \( T \). That is, one basis is \( \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \). If you don’t like this proof, you can use the matrix of \( T \) found in part (c) to do the question.

(b) Find a basis for the range of \( T \).

**Solution:** Since the kernel is 2-dimensional, the range must be 1-dimensional. To find it, we can use any nonzero vector in the range as a basis vector. If we calculate \( T(e_1) \) we get \( \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \), so this vector spans the range.

(c) Find the matrix of the transformation.

**Solution:** We have already found \( T(e_1) \), we now calculate \( T(e_1) \) and \( T(e_1) \), and put these into a matrix to get \( [T] = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} \).
2. Let $T : F^{2 \times 2} \to F^{2 \times 2}$ be defined by $T(A) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} A - A \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(a) Find a basis for the kernel of $T$.

**Solution:** One approach is to find a simple formula for $T$ in terms of the entries in a matrix. That is, $T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} c - b & d - a \\ a - d & b - c \end{pmatrix}$. To be in the kernel, we need $b = c$ and $a = d$ so a basis for the kernel is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

(b) Find a basis for the range of $T$.

**Solution:** If $T(A) = B$, where $B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ then by the formula above, we need $w, x, y, z$ so that $c - b = w$, $d - a = x$, $a - d = y$, $b - c = z$ to have a solution. The system is consistent when $z = -w$, $y = -x$, leading to a basis for the range of $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$.

(c) Find a simple formula for $T^2 = T \circ T$.

**Solution:** Let $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Symbolically, $(T \circ T)(A) = T(T(A)) = T(BA - AB) = B(BA - AB) - (BA - AB)B = B^2A - 2BAB + AB^2$. If we want a formula as in part (a), we have $T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a - 2d & 2b - 2c \\ 2c - 2b & 2d - 2a \end{pmatrix}$.

3. Let $T : P^3 \to R^{2 \times 2}$ be defined by $T(p(x)) = p(A)$, where $A = \begin{pmatrix} 2 & 4 \\ -1 & 0 \end{pmatrix}$. For example, $T(x^2 - 2x + 3) = A^2 - 2A + 3I = \begin{pmatrix} 0 & 8 \\ -2 & -4 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ -1 & 0 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -1 & -1 \end{pmatrix}$.

(a) Find a basis for the kernel of $T$.

**Solution:** As in the previous problem, one approach is to find a formula for $T$ in terms of the coefficients of $p(x)$. If $p(x) = ax^3 + bx^2 + cx + d$ then $T(p(x)) = aA^3 + bA^2 + cA + dI = a \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix} + b \begin{pmatrix} 0 & 8 \\ -2 & -4 \end{pmatrix} + c \begin{pmatrix} 2 & 4 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. To get 0, we need $-8a + 2c + d = 0$, $8b + 4c = 0$, $-2b - c = 0$, $-8a - 4b + d = 0$. If we
take $a$ and $b$ as free variables then we need $d = 8a + 4b$, $c = -2b$. Polynomials in the kernel have form $ax^3 + bx^2 - 2bx + 8a + 4b$, with basis $\{x^3 + 8, x^2 - 2x + 4\}$.

(b) Find a basis for the range of $T$.

**Solution:** One approach is to apply to $T$ to the standard basis polynomials $1, x, x^2, x^3$, to product a spanning set, and then remove dependent vectors. In this case, we get the matrices $I, A, A^2, A^3$. Since $A^3 = -8I$ and $A^2 = 2A - 4I$, a basis is $I, A$.

For extra credit:

4. The matrix in problem 1 had 0 for a determinant. Did this have a big influence on the problem? That is, if a matrix $A$ with nonzero determinant had been used, how much different in character would the problem have been?

**Solution:** If $A$ is invertible then the rank would be 3 and the nullity would be 0. The reason is this: To be in the kernel of the transformation we need $\det(A_1(v)) = \det(A_2(v)) = \det(A_3(v)) = 0$. Letting the columns of $A$ be $v_1, v_2, v_3$, if $T(v) = 0$ then $v$ must be a combination of $v_2$ and $v_3$ for the $x$-coordinate to be 0, and so on. This means $v = c_1v_2 + c_2v_3$, $v = d_1v_1 + d_2v_3$, and $v = e_1v_1 + e_2v_2$. Subtracting the first two gives $-d_1v_1 + c_1v_2 + (c_2 - d_2)v_3 = 0$. By independence of the columns, $c_1 = d_1 = 0$ and $c_2 = d_2$. But now $v = c_2v_3 = e_1v_1 + e_2v_2$ and independence shows all the $c's$, $d's$, $e's$ are 0, so $v = 0$ only.

5. More generally, if $A$ is $n \times n$ and $T : F^n \rightarrow F^n$ is defined analogously to what was done in problem 1, how does $T$ vary with the rank of $A$?

**Solution:** This is like a homework problem in the last homework set. If $A$ has rank $n - 2$ or less, then $\det(A_i(v)) = 0$ for all $i$, since $v$ can at most replace one of two dependent columns. Thus, $T$ is the 0-transformation. If $A$ has rank $n - 1$ then $T$ has rank 1 and if $A$ is invertible then $T$ has rank $n$. 

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6. What are the range and kernel of $T$ in problem 3 if the domain is $P^n$ instead of $P^3$?

**Solution:** It turns out that for all $n \geq 2$, $A^n$ is a combination of $I$ and $A$. This follows by induction on $n$, the inductive step going like this: If $A^n = aI + bA$ then $A^{n+1} = aA + bA^2 = -4bI + (a + 2b)A$. This means that the rank of $T$ is 2 for all $n \geq 1$, with basis $\{I, A\}$. Thus, the nullity is $n + 1 - 2 = n - 1$. There must be $n - 1$ independent polynomials $p(x)$ for which $p(A) = 0$. In fact, give that $x^2 - 2x + 4$ is in the kernel it follows that any multiple of this polynomial is also in the kernel. Thus, a basis is $\{x^2 - 2x + 4, x(x^2 - 2x + 4), \ldots, x^{n-2}(x^2 - 2x + 4)\}$. 

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