From the book: 3.5.2 (Sections 3.5).

Problem 3.5.2 For each of the following matrices, determine all real eigenvalues, all eigenspaces, and if the matrix is diagonalizable. If it is diagonalizable, show the change of basis making it diagonal.

(a) \( A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \)

Solution: \[
\begin{vmatrix} x - 3 & -1 & 0 \\ -1 & x - 2 & 0 \\ -1 & -1 & x \end{vmatrix} = x((x-3)(x-2)-1) = x(x^2-5x+5). \]
The eigenvalues are 0 and \( \frac{5 \pm \sqrt{5}}{2} \). The eigenspace of 0 is spanned by \( e_3 \). We have

\[
A - \frac{5 + \sqrt{5}}{2} I = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 & 0 \\ 1 & \frac{1-\sqrt{5}}{2} & 0 \\ 1 & 1 & -5+\frac{\sqrt{5}}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -\frac{1+\sqrt{5}}{2} \ 0 & 1 & -\frac{5+\sqrt{5}}{2} \ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -\sqrt{5} \\ 0 & 1 & -\frac{5-\sqrt{5}}{2} \\ 0 & 0 & 0 \end{pmatrix}
\]

A basis for the eigenspace of \( \frac{5 + \sqrt{5}}{2} \) is \( \begin{pmatrix} 2\sqrt{5} \\ 5 - \sqrt{5} \\ 2 \end{pmatrix} \). A trick for \( \frac{5 - \sqrt{5}}{2} \) is to take

the (algebraic) conjugate of everything. A basis for its eigenspace is \( \begin{pmatrix} -2\sqrt{5} \\ 5 + \sqrt{5} \\ 2 \end{pmatrix} \).

The matrix is diagonalizable, with the three eigenvectors as the change of basis.

(b) \( B = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix} \)

Solution: \[
\begin{vmatrix} x - 1 & 0 & -2 \\ -1 & x - 1 & 1 \\ 1 & 0 & x - 2 \end{vmatrix} = (x-1)((x-1)(x-2)+2) = (x-1)(x^2-3x+4).
\]
The only real eigenvalue is 1, the matrix is not diagonalizable (over the reals), and the eigenspace of 1 is spanned by \( e_2 \).
(c) \[ C = \begin{pmatrix} 0 & r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

**Solution:** The characteristic polynomial is \( x^4 \), there is only one eigenvalue, 0, and the eigenspace of 0 is 1, 2, 3, or 4-dimensional depending on how many of \( r, s, t \) are 0: \( e_1 \) is always an eigenvector, and \( e_2 \) will also be one if \( r = 0 \), \( e_3 \) is an eigenvector if \( s = 0 \), and \( e_4 \) is an eigenvector if \( t = 0 \). The only time \( C \) is diagonalizable is if all three are 0, in which case it is already diagonal.

(d) \[ D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \]

**Solution:** The characteristic polynomial is \((x - 1)(x - 2)(x - 3)(x - 4)\), with four distinct eigenvalues, 1, 2, 3, 4. The eigenspaces are spanned by \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 6 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 32 \\ 24 \\ 12 \\ 3 \end{pmatrix}, \) respectively. The matrix is diagonalizable, with these four vectors as the change of basis.

(e) \[ E = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

**Solution:** The characteristic polynomial is \((x - 2)(x - 1)^3\), the eigenspace of 2 is spanned by \begin{pmatrix} 1 \\ 1 \\ 5 \\ 0 \end{pmatrix}, and the eigenspace of 1 is spanned by \( e_3, e_4 \). There are only three independent eigenvalues so \( E \) is not diagonalizable.
Also do the following:

1. Evaluate \( \lim_{n \to \infty} A^n \) where \( A = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \).

**Solution:** If \( A = PDP^{-1} \) then \( A^n = PD^nP^{-1} \) and \( \lim_{n \to \infty} A^n = P \lim_{n \to \infty} D P^{-1} \). So the first step is to find \( P \) and \( D \).

We have \( \begin{vmatrix} x - 2/3 & -1/3 \\ -1/3 & x - 2/3 \end{vmatrix} = x^2 - \frac{4}{3}x + \frac{1}{3} = (x-1)(x-\frac{1}{3}) \).

We can use \( P = \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{pmatrix} \) with inverse \( \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \) and \( D = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \). Since \( D^n \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), \( A^n \to \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \).

2. A linear operator \( T : V \to V \) is called a reflection if \( T^2 = T \circ T = I \). In this problem we prove that all reflections are diagonalizable.

   (a) Showing that the only possible eigenvalues of \( T \) are 1 and -1.

   **Solution:** Let \( T(v) = cv \). Then \( v = Iv = T^2(v) = T(T(v)) = T(cv) = c^2v \). That is, \( c^2 = 1 \) so the only possible values for \( c \) are 1 and -1.

   (b) If \( T \) has just a single eigenvalue, prove that \( T \) is diagonalizable.

   **Solution:** The trick is to look at \( T^2 - I = (T-I)(T+I) \). If \( T \) has a single eigenvalue, say 1, then -1 is not an eigenvalue, so \( T+I \) has nullity 0. Now suppose that \( v \) is any vector in \( V \). Then \( T^2(v) = v \), so \( (T+I)(T-I)(v) = T^2(v) - I(v) = 0 \). If \( (T-I)(v) = w \), then \( (T+I)(w) = 0 \). Thus, \( w \) is in the kernel of \( T+I \). Since \( T+I \) is invertible, this can only happen if \( w = 0 \). Thus, \( (T-I)(v) = 0 \), meaning that \( T(v) = v \) for all \( v \in V \). That is, \( T \) is the identity transformation, and any basis for \( V \) gives \( \begin{bmatrix} T \end{bmatrix} = I \), the identity matrix, a diagonal matrix.

   If \( T \) has only -1 as an eigenvalue, the proof is similar. This time, \( T-I \) is invertible. Again, for any \( v \), \( (T^2 - I)(v) = 0 \), which we write \( (T-I)(T+I)(v) \), and we let \( (T+I)(v) = w \). Now \( (T-I)(w) = 0 \), implying \( w = 0 \), so \( (T+I)(v) = 0 \), meaning \( T(v) = -v \) for all \( v \). Thus, for any basis, \( \begin{bmatrix} T \end{bmatrix} = -I \), again a diagonal matrix.
(c) If $v$ is any vector in $V$, show that $v + T(v)$ is an eigenvector with eigenvalue 1.

**Solution:** Let $u = v + T(v)$. Then $T(u) = T(v + T(v)) = T(v) + T^2(v) = T(v) + Iv = v + T(v) = u$. That is, $T(u) = u$, so $u$ is an eigenvector with eigenvalue 1.

(d) If $v$ is any vector in $V$, give an expression like the one in part (c) for an eigenvector with eigenvalue -1.

**Solution:** Use $v - T(v)$. If $w = v - T(v)$ then $T(w) = T(v - T(v)) = T(v) - v = -w$.

(e) Finally, if $T$ has two eigenvalues, with eigenspaces $E_{-1}$ and $E_1$, prove that $V = E_{-1} \oplus E_1$. Hint: If $v$ is any vector in $V$, show that $v$ is the sum of something in $E_{-1}$ and something in $E_1$.

**Solution:** If $v$ is any vector in $V$ then $v = \frac{1}{2}(v + T(v)) + \frac{1}{2}(v - T(v))$, with the first term in $E_1$ and the second in $E_{-1}$. Thus, $V = E_1 + E_{-1}$, and the intersection of eigenspaces is 0 so $V$ is the direct sum, showing that there are enough eigenvectors to diagonalize $T$.

3. Let $T : F^{n \times n} \to F^{n \times n}$ be defined by $T(A) = BA$, where $B$ is some fixed matrix.

(a) If $v$ is an eigenvector of $B$, show that $A = (v | 0 | \cdots | 0)$ is an eigenvector of $T$.

**Solution:** If $Bv = cv$ then $T(A) = BA = B(v | 0 | \cdots | 0) = (Bv | B0 | \cdots | B0) = (cv | 0 | \cdots | 0) = cA$.

(b) Prove that $c$ is an eigenvalue for $B$ if and only if $c$ is an eigenvalue for $T$.

**Solution:** Part (a) showed that if $c$ is an eigenvalue for $B$, then it is an eigenvalue for $T$ as well. Suppose that $c$ is an eigenvalue for $T$. Then for some $A \neq 0$, $T(A) = cA$. That is, $BA = cBA$. This means that if $v$ is any column of $A$, then $Bv = cv$. If $A \neq 0$ then $A$ has a nonzero column to use for $v$.

(c) Find the characteristic polynomial of $T$. It should have degree $n^2$.

**Solution:** We find the matrix of $T$ with respect to a nice basis. The usual basis for $F^{n \times n}$ is $\{E_{1,1}, E_{1,2}, \ldots, E_{1,n}, E_{2,1}, \ldots, E_{n,n}\}$ but it turns out that there is a nicer basis for this problem: $\{E_{1,1}, E_{2,1}, \ldots, E_{n,1}, E_{1,2}, \ldots, E_{n,n}\}$. Let $B = (v_1 | v_2 | \cdots | v_n)$. Then $BE_{i,j}$ is the matrix that has $v_i$ in column $j$ and 0’s for all the other columns. What this means is that with respect to this basis, the
Matrix of $T$ will be block diagonal with $B$’s on the diagonal. The characteristic polynomial of this matrix is $c_B(x)^n$.

As an example, let $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. The good basis to use is $\{E_{1,1}, E_{2,1}, E_{1,2}, E_{2,2}\}$.

We have $BE_{1,1} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$, $BE_{2,1} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}$, $BE_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$, $BE_{2,2} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}$ . Converting to coordinates, $[T] = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}$, and the characteristic polynomial is $((x - 1)(x - 4) - 6)^2$.

(d) Prove that $T$ is diagonalizable if and only if $B$ is diagonalizable.

**Solution:** Suppose that $B$ is diagonalizable, with $n$ linearly independent eigenvectors $u_1, u_2, \ldots, u_n$. Let $A_{i,j}$ be the matrix that has $u_i$ for its $j$’th column and 0’s for the other columns. Then $BA_{i,j} = c_iA_{i,j}$ so all these matrices are eigenvectors for $T$. Moreover, these matrices form a linearly independent set (ideally one should say why!), so $T$ has $n^2$ linearly independent eigenvectors, showing that $T$ is diagonalizable.

As an alternative, $T$ is diagonalizable if and only if $[T]$ is diagonalizable. If we write $[T]$ as above, as a block diagonal matrix (often written $\text{Diag}(B, B, \ldots, B)$), then $Q = \text{Diag}(P, P, \ldots, P)$ will diagonalize $[T]$, where $P$ diagonalizes $B$. That is, if $P^{-1}BP = D$, a diagonal matrix, then

$$Q^{-1}[T]Q = \text{Diag}(P^{-1}, P^{-1}, \ldots, P^{-1}) \text{Diag}(B, B, \ldots, B)\text{Diag}(P, P, \ldots, P)$$

$$= \text{Diag}(P^{-1}BP, P^{-1}BP, \ldots, P^{-1}BP)$$

$$= \text{Diag}(D, D, \ldots, D),$$

which is diagonal.

We still have to prove that if $T$ is diagonalizable, then $B$ is also diagonalizable. This seemed to be hard, so I considered it an extra credit problem. I gave a complicated solution: Suppose that $c$ is an eigenvalue of $B$ with an eigenspace of dimension $k$. I claim the eigenspace of $T$ has dimension $kn$. Thus, if the algebraic multiplicity of $c$ is the same as its geometric multiplicity for $T$, it will be for $B$, as well. All we really need for this direction of the proof is to show that the geometric multiplicity of $c$ can’t be larger than $kn$ for $T$. By parts (a) and (b), every eigenvector of $T$ with eigenvalue $c$ is a matrix, all of whose columns are eigenvectors of $B$. I claim that any $kn + 1$ matrices that are eigenvectors
for \(T\) must be linearly dependent. To that end, suppose that \(A_1, \ldots, A_{kn+1}\) are all eigenvectors for \(T\). Every column in every \(A\) is an eigenvector for \(B\), so any \(k+1\) column vectors must be linearly dependent. Thus, for any \(j > k\), the first columns of \(A_1, A_2, \ldots, A_k, A_j\) form a dependent set. This means some combination of these matrices will have form \((0|u_2| \cdots |u_n)\). We can do this for each \(j > k\), giving us \(kn + 1 - k = k(n - 1) + 1\) such matrices. We now have \(k(n - 1) + 1\) matrices all of which have 0 for a first column. We form combinations of these as before, to produce \(k(n - 2) + 1\) matrices with first two columns equal to 0. A proper proof would be by induction, but as you see, we can continue this till we get to one matrix \((k(n - n) + 1)\), all of whose columns are 0. That is, some combination of combinations of combinations \ldots gives the 0 matrix. A combination of combinations is a combination, so we produce a dependence relation in this manor.

The solution from the class was easier: If \(T\) is diagonalizable, then \(T\) has \(n^2\) linearly independent eigenvectors. Each eigenvector of \(T\) is an \(n \times n\) matrix, consider the first column from each of these matrices. There are \(n^2\) of them, and each is an eigenvector for \(B\). What we need is for \(n\) of these to be independent. We use the fact that every spanning set contains a basis, and show that these \(n^2\) vectors span \(F^n\). Since the \(n^2\) matrices span \(F^{n \times n}\), given any vector \(v \in F^n\), consider the matrix \((v|0| \cdots |0)\). This matrix must be a combination of the \(n^2\) matrices, which means that \(v\) is a combination of the columns of those matrices. Thus, every vector in \(F^n\) is a combination of the first columns, showing these span \(F^n\), so there must be \(n\) linearly independent vectors among them.

For extra credit:

4. Let \(T : F^{n \times n} \rightarrow F^{n \times n}\) be defined by \(T(A) = BA - AB\), where \(B\) is some fixed matrix. Prove or give a counterexample: \(T\) is diagonalizable if and only if \(B\) is diagonalizable. I will give partial credit if you can give examples of eigenvalues and eigenvectors for \(T\).

**Solution:** I will only give a partial solution here. It turns out that \(T\) is diagonalizable if and only if \(B\) is. I will only talk about the case where \(B\) is diagonalizable. In that case, we try to get \(n^2\) independent eigenvectors (matrices) for \(T\). Since \(B\) is diagonalizable, there are \(n\) linearly independent eigenvectors \(v_1, v_2, \ldots, v_n\) for \(B\) with eigenvalues \(c_1, c_2, \ldots, c_n\), not all necessarily distinct. It takes a little effort, but if \(B\) is diagonalizable, then \(B^t\) is also diagonalizable, so \(B^t\) also has \(n\) linearly independent eigenvectors \(w_1, \ldots, w_n\). Let these vectors have eigenvalues \(d_1, \ldots, d_n\). These vectors have the property that \(B^tw = dw\), or, taking transposes, \(w^tB = dw^t\).
Now let \( A_{i,j} = v_i w_j^t \). Each \( A_{i,j} \) is an \( n \times n \) matrix. Moreover,
\[
T(A_{i,j}) = Bv_i w_j^t - v_i w_j^t B = c_i v_i w_j^t - d_j v_i w_j^t = (c_i - d_j)v_i w_j^t.
\]
That is, each \( A_{i,j} \) is an eigenvector, with eigenvalue \( c_i - d_j \). It turns out that the \( A \)'s are all linearly independent (I will leave this to you to check). Consequently, we have \( n^2 \) independent eigenvectors for \( T \), so \( T \) is diagonalizable.