From the book:

**Problem 4.1.2** If $S$ is a subset of an inner product space $V$, define $S^\perp$ to be the set

$$S^\perp = \{ v \in V \mid \langle u, v \rangle = 0 \text{ for all } u \in S \}.$$

(a) Prove that $S^\perp$ is a subspace of $V$.

**Solution:** We use the subspace test. Note that $S^\perp$ is nonempty since it contains the 0-vector. If $u$ and $v$ are $S^\perp$ and $s$ is any element of $S$ then $\langle u+v, s \rangle = \langle u, s \rangle + \langle v, s \rangle = 0 + 0 = 0$, so $u+v$ is orthogonal to all elements of $S$. That is, $u+v \in S^\perp$. Similarly, $\langle cu, s \rangle = c \langle u, s \rangle = 0$ so $cu \in S^\perp$. Thus, $S^\perp$ satisfies the subspace test conditions.

(b) Find $S^\perp$ if $S = \{(1, 1, 1), (2, 1, 0)\}$ (using the dot product).

**Solution:** Given $(x, y, z) \in S^\perp$, we need $\langle (x, y, z), (1, 1, 1) \rangle = 0$ and $\langle (x, y, z), (2, 1, 0) \rangle = 0$. That is, we need $x + y + z = 0$, $2x + y = 0$. If we let $x$ be free then $y = -2x$, $z = -x - y = x$. Thus, $S^\perp$ is the set of points of the form $(x, -2x, x)$, a set with basis $(1, -2, 1)$.

(c) Find $S^\perp$ if $S = \{(1, 1, 1), (2, 1, 0), (1, 0, -1)\}$ (using the dot product).

**Solution:** In addition to the two equations in part (b), we also need $x - z = 0$. Since this is satisfied by $(1, -2, 1)$, the solution here is the same as in part (b). That is, $S^\perp = \text{Span}\{(1, -2, 1)\}$.

**Problem 4.1.16** Prove the polar identities in an inner product space:

(a) For real spaces, $\langle u, v \rangle = \frac{1}{4}||u+v||^2 - \frac{1}{4}||u-v||^2$.

**Solution:** We have

$$\frac{1}{4}||u+v||^2 - \frac{1}{4}||u-v||^2 = \frac{1}{4}(\langle u + v, u + v \rangle - \langle u - v, u - v \rangle)$$

$$= \frac{1}{4}(\langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle - \langle u, u \rangle + 2\langle u, v \rangle - \langle v, v \rangle)$$

$$= \frac{1}{4}(4\langle u, v \rangle) = \langle u, v \rangle.$$
(b) For complex spaces, $\langle u, v \rangle = \frac{1}{4}||u + v||^2 + \frac{1}{4}i||u + iv||^2 - \frac{1}{4}||u - v||^2 - \frac{1}{4}i||u - iv||^2$.

**Solution:**

\[
\frac{1}{4}||u + v||^2 + \frac{1}{4}i||u + iv||^2 - \frac{1}{4}||u - v||^2 - \frac{1}{4}i||u - iv||^2
\]

\[
= \frac{1}{4}(\langle u + v, u + v \rangle + i\langle u + iv, u + iv \rangle - \langle u - v, u - v \rangle - i\langle u - iv, u - iv \rangle)
\]

\[
= \frac{1}{4}(\langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle - \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle - \langle v, v \rangle)
\]

\[
+ \frac{1}{4}(i\langle u, u \rangle - \langle v, u \rangle + \langle v, v \rangle - i\langle u, u \rangle - \langle v, u \rangle + \langle v, v \rangle - i\langle v, v \rangle)
\]

\[
= \frac{1}{4}(4\langle u, v \rangle) = \langle u, v \rangle.
\]

A solution from the class that I liked: $\langle u + v, u + v \rangle = ||u||^2 + ||v||^2 + 2\text{Re}(u,v)$. Thus, $||u + v||^2 - ||u - v||^2 = 4\text{Re}(u,v)$. Now if we replace $v$ by $iv$, then $\text{Re}(u,iv) = \text{Re}(-i\langle u, v \rangle) = \text{Im}(u,v)$. This means that $||u + iv||^2 - ||u - iv||^2 = 4\text{Im}(u,v)$. Thus,

\[
\frac{1}{4}||u + v||^2 + \frac{1}{4}i||u + iv||^2 - \frac{1}{4}||u - v||^2 - \frac{1}{4}i||u - iv||^2
\]

\[
= \frac{1}{4}(4\text{Re}(u,v) + 4i\text{Im}(u,v))
\]

\[
= \langle u, v \rangle.
\]

**Problem 4.2.2** Let $W = \text{Span}\{v_1, v_2, v_3\} = \text{Span}\{(1,1,0,0), (1,0,1,0), (1,0,0,1)\}$

(a) Find an orthogonal basis for $W$.

**Solution:** We apply Gram-Schmidt: Let the orthogonal basis be $\{u_1, u_2, u_3\}$. We let $u_1 = v_1 = (1,1,0,0)$. Next, $u_2 = v_2 - \text{Proj}_{u_1}(v_2)u_1 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle}u_1 = (1,0,1,0) - \frac{1}{2}(1,1,0,0) = \left(\frac{1}{2}, -\frac{1}{2}, 1, 0\right)$. To avoid fractions, let’s use twice this, $u_2 = (1,-1,2,0)$. Finally,

\[
u_3 = v_3 - \text{Proj}_{u_1}(v_3)u_1 - \text{Proj}_{u_2}(v_3)u_2 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle}u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle}u_2
\]

\[= (1,0,0,1) - \frac{1}{2}(1,1,0,0) - \frac{1}{6}(1,-1,2,0) = \left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1\right)
\]

We can multiply this by 3 to get an orthogonal basis

\[
\{(1,1,0,0), (1,-1,2,0), (1, -1, -1, 3)\}.
\]
(b) Find a basis for \( W^\perp \).

**Solution:** We need vectors perpendicular to all three of the basis vectors above. That is, we solve the system of equations \( w + x = 0, \ w - x + 2y = 0, \ w - x - y + 3z = 0 \). If we let \( w \) be free then \( x = -w, \ y = -w, \ z = -w \) leading to the basis \( \{(1, -1, -1, -1)\} \).

**Problem 4.2.8** If \( \{u_1, u_2, \ldots, u_n\} \) is an orthonormal basis of an inner product space \( V \), show that for any \( v \in V \), \( ||v||^2 = \langle v, u_1 \rangle^2 + \langle v, u_2 \rangle^2 + \cdots + \langle v, u_n \rangle^2 \).

**Solution:** I think \( V \) has to be a real inner product space here. Let \( v \in V \). We can write \( v \) as a combination of basis vectors, \( v = c_1 u_1 + \cdots + c_n u_n \). By Theorem 2.2.2 in the book, \( c_1 = \langle v, u_1 \rangle, \ldots, c_n = \langle v, u_n \rangle \). We have

\[
||v||^2 = \langle v, v \rangle = \langle v, c_1 u_1 + c_2 u_2 + \cdots + c_n u_n \rangle \\
= c_1 \langle v, u_1 \rangle + c_2 \langle v, u_2 \rangle + \cdots + c_n \langle v, u_n \rangle \\
= c_1 c_1 + c_2 c_2 + \cdots + c_n c_n.
\]

That is, \( ||v||^2 = ||\langle v, u_1 \rangle||^2 + ||\langle v, u_2 \rangle||^2 + \cdots + ||\langle v, u_n \rangle||^2 \). If we are in a real inner product space, we can get rid of the absolute values.

**Problem 4.2.10** Let \( L \) be the line spanned by a nonzero vector \( v \in \mathbb{R}^n \). Let \( w \in \mathbb{R}^n \).

(a) Show that the distance between \( w \) and \( L \) is \( ||w - \text{proj}_v(w)|| \).

**Solution:** This is like a problem we did in class. The line is \( L(t) = tv \), and the distance from \( w \) to the line means the smallest distance, that is, the smallest \( \text{dist}(tv, w) \) can be. Now \( \text{dist}(u, v) = ||u - v|| \). In our case, this is \( ||tv - w|| \). As in class, we can minimize the square. We have

\[
||tv - w||^2 = \langle tv - w, tv - w \rangle = t^2 \langle v, v \rangle - 2t \langle v, w \rangle + \langle w, w \rangle,
\]

and this will take on a minimum when \( t = \frac{\langle u, v \rangle}{\langle v, v \rangle} \) (I differentiated, set the expression equal to 0 and solved for \( t \)). Plugging this \( t \) value in, we have Distance = \( ||tv - w|| = ||w - tv|| = ||w - \text{proj}_v(w)|| \).
(b) Generalize part (a) and describe how, using the Gram-Schmidt theorem, to find the distance between a point \( w \in \mathbb{R}^n \) and the subspace \( \text{Span}\{v_1, v_2, \ldots, v_s\} \).

**Solution:** The geometric interpretation of the above is that you want the perpendicular distance from \( w \) to the line, and \( w - \text{proj}_v(w) \) is the projection of \( w \) perpendicular to \( v \), so we want the length of this vector. For a general subspace \( V \), we still want the perpendicular distance to the subspace which is the length of the projection of \( w \) orthogonal to the space. That is, if we have a basis for \( V \), and we use Gram-Schmidt to get an orthogonal basis, \( \{u_1, u_2, \ldots, u_k\} \), then the distance to the subspace is \( ||w - \text{proj}_{u_1}(w) - \text{proj}_{u_2}(w) - \cdots - \text{proj}_{u_k}(w)|| \).

Alternatively, knowing the answer, it is easy to prove it is correct, without Hessians, without calculus. The proof goes like this: We let \( V \) be the space, and use an orthogonal basis, \( \{u_1, u_2, \ldots, u_k\} \). We want to minimize \( ||w - v|| \) where \( v \in V \), and we do this using the following trick: First, as usual, we minimize the square of the distance. Next, \( w = w - \text{proj}_V(w) + \text{proj}_V(w) \). In class we’ve seen several times that \( w - \text{proj}_V(w) \) is orthogonal to each of \( u_1, \ldots, u_k \). If we write \( v = c_1 u_1 + \cdots + c_k u_k \), then

\[
||w - v||^2 = ||(w - \text{proj}_V(w)) + \left( \frac{\langle w, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \cdots - \frac{\langle w, u_k \rangle}{\langle u_k, u_k \rangle} u_k \right) + c_1 u_1 + \cdots + c_k u_k ||^2
\]

\[
= ||w - \text{proj}_V(w)||^2 + \left| \left| \frac{\langle w, u_1 \rangle}{\langle u_1, u_1 \rangle} \right| - c_1 \right| ||u_1||^2 + \cdots + \left| \left| \frac{\langle w, u_k \rangle}{\langle u_k, u_k \rangle} \right| - c_k \right| ||u_k||^2.
\]

To minimize this quantity, we need each \( c_i = \frac{\langle w, u_i \rangle}{\langle u_i, u_i \rangle} \), zeroing all these extra terms.

(c) Find the distance between the point \((1, 1, 1, 0)\) and \( V = \text{Span}\{(2,0,0,1), (1,1,0,0)\} \).

**Solution:** First, we make the basis orthogonal. We let \( u_1 = (2, 0, 0, 1) \) and \( u_2 \) be an appropriate multiple of

\[
(1, 1, 0, 0) - \frac{\langle (1, 1, 0, 0), (2, 0, 0, 1) \rangle}{\langle (2, 0, 0, 1), (2, 0, 0, 1) \rangle} (2, 0, 0, 1) = \left( \frac{1}{5}, 1, 0, - \frac{2}{5} \right).
\]

That is, we have orthogonal basis \( \{(2, 0, 0, 1), (1, 5, 0, -2)\} \). The two projections of \( w \) onto these vectors are \( \frac{2}{5} (2, 0, 0, 1) \) and \( \frac{6}{30} (1, 5, 0, -2) \). The distance we seek is

\[
|| (1, 1, 1, 0) - \frac{2}{5} (2, 0, 0, 1) - \frac{1}{5} (1, 5, 0, -2) || = || (0, 0, 1, 0) || = 1.
\]
Also do the following problems.

1. Using the Euclidean inner product, find the standard matrix for the following orthogonal projections.

   (a) The projection of $R^2$ onto the line $y = 5x$.

   **Solution:** The line is spanned by $v = (5, 1)$ so $T = \text{proj}_v$. That is,
   
   $T(x, y) = \frac{\langle (x, y), (5, 1) \rangle}{\langle (5, 1), (5, 1) \rangle} (5, 1) = \frac{5x + y}{26} (5, 1),$
   
   which has matrix $A = \begin{pmatrix} \frac{25}{26} & \frac{5}{26} \\ \frac{5}{26} & \frac{1}{26} \end{pmatrix}$. As a check, note that $A^2 = A$.

   (b) The projection of $R^3$ onto the line spanned by (1, 1, 1).

   **Solution:**
   
   $T(x, y, z) = \frac{\langle (x, y, z), (1, 1, 1) \rangle}{\langle (1, 1, 1), (1, 1, 1) \rangle} (1, 1, 1) = \frac{x + y + z}{3} (1, 1, 1),$
   
   with matrix $\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$.

   (c) The orthogonal projection of $R^3$ onto the plane $x + y - 2z = 0$.

   **Solution:** First we get an orthogonal basis for the plane, then we add the projections onto the two basis vectors. To get an orthogonal basis, you can start with any basis and apply Gram-Schmidt. Alternatively, note that $u = (1, -1, 0), v = (1, 1, 1)$ are orthogonal vectors on the plane. We have
   
   $T(x, y, z) = \frac{\langle (x, y, z), (1, -1, 0) \rangle}{\langle (1, -1, 0), (1, -1, 0) \rangle} (1, -1, 0) + \frac{\langle (x, y, z), (1, 1, 1) \rangle}{\langle (1, 1, 1), (1, 1, 1) \rangle} (1, 1, 1)$
   
   $= \frac{x - y}{2} (1, -1, 0) + \frac{x + y + z}{3} (1, 1, 1),$
   
   with matrix $\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix} + \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 5/6 & -1/6 & 1/3 \\ -1/6 & 5/6 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$. 

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2. Define \( \langle p(x), q(x) \rangle \) on \( P^2 \) by \( \langle p(x), q(x) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) \).

(a) Prove that \( \langle p(x), q(x) \rangle \) is an inner product.

**Solution:** First, \( \langle q(x), p(x) \rangle = q(-1)p(-1) + q(0)p(0) + q(1)p(1) = p(-1)q(-1) + p(0)q(0) + p(1)q(1) = \langle p(x), q(x) \rangle \). For the linearity conditions, \( \langle cp(x), q(x) \rangle = cp(-1)q(-1) + cp(0)q(0) + cp(1)q(1) = c(p(-1)q(-1) + p(0)q(0) + p(1)q(1)) = c\langle p(x), q(x) \rangle \) and

\[
\langle p(x) + r(x), q(x) \rangle = \langle p(x) + r(x), q(x) \rangle = \langle p(-1) + r(-1) \rangle q(-1) + (p(0) + r(0))q(0) + (p(1) + r(1))q(1) = \langle p(x), q(x) \rangle + \langle r(x), q(x) \rangle.
\]

Finally, \( \langle p(x), p(x) \rangle = p(-1)^2 + p(0)^2 + p(1)^2 \geq 0 \). For the expression to be 0, we need \( p(-1) = p(0) = p(1) = 0 \). If we let \( p(x) = ax^2 + bx + c \), we need \( a-b+c = 0, c = 0, a+b+c = 0 \). For these to be satisfied, we need \( a = b = c = 0 \), so \( p(x) = 0 \).

(b) Find an orthogonal basis for \( P^2 \) with respect to this inner product.

**Solution:** We use Gram-Schmidt. If we start with the standard basis, \( \{1, x, x^2\} \) and convert to \( \{u, v, w\} \), then we can take \( u = 1 \). Next, \( x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x - v \), and \( x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle x^2, x \rangle}{\langle x, x \rangle} x = x^2 - \frac{2}{3} \), so an orthogonal basis is \( \{1, x, 3x^2 - 2\} \).

(c) Verify the Cauchy-Schwarz inequality for this inner product when \( p(x) = x^2 + x + 1, \ q(x) = x^2 + x \).

**Solution:** Cauchy-Schwarz: \( \langle u, v \rangle^2 \leq \langle u, u \rangle \langle v, v \rangle \). In our case, \( \langle p(x), q(x) \rangle = 0 + 0 + 6 = 6, \ \langle p(x), p(x) \rangle = 1 + 1 + 9 = 11, \ \langle q(x), q(x) \rangle = 0 + 0 + 4 = 4 \). Since \( 6^2 = 36 \) and \( 11 \times 4 = 44 \), the inequality is satisfied.

(d) Find a formula for the orthogonal projection of \( P^2 \) onto \( V \), the space spanned by \( x \).

**Solution:** The formula is \( T(p(x)) = \text{proj}_x(p(x)) \). If we let \( p(x) = ax^2 + bx + c \) we have

\[
T(p(x)) = \frac{\langle ax^2 + bx + c, x \rangle}{\langle x, x \rangle} x = \frac{(a-b+c)(-1)+0+a+b+c}{2} x = bx.
\]
(e) Find a formula for the orthogonal projection of $P^2$ onto $P^1$.

**Solution:** We add the projections onto 1 and onto $x$. We calculated the projection onto $x$ to be $bx$. The projection onto 1 is

$$\frac{\langle ax^2 + bx + c, 1 \rangle}{\langle 1, 1 \rangle} = \frac{a - b + c + a + b + c}{3} = \frac{2a + 3c}{3}.$$  

A formula for the transformation is $T(ax^2 + bx + c) = bx + \frac{1}{3}(2a + 3c)$. As a quick check, $T$ should be the identity if restricted to $P^1$. When $a = 0$ we get $T(bx + c) = bx + c$, as we should.

(f) Determine whether $\langle p(x), q(x) \rangle$ is an inner product on $P^3$.

**Solution:** This is no longer an inner product on $P^3$. The first 3 properties along with nonnegativity still hold. However, if $p(x) = ax^3 + bx^2 + cx + d$ then $\langle p(x), p(x) \rangle = 0$ provided $-a + b - c + d = 0$, $d = 0$, $a + b + c + d = 0$, and this has, as one solution, $p(x) = x^3 - x$. That is, $\langle x^3 - x, x^3 - x \rangle = 0$, violating the strict positivity of $\langle p(x), p(x) \rangle$ when $p(x) \neq 0$.

For extra credit:

3. The transformation $T(x, y) = (2x - y, 2x - y)$ is a projection of $R^2$ onto the line $y = x$. This projection is not the orthogonal projection with respect to the usual inner product. Is it the orthogonal projection of $R^2$ onto $y = x$ with respect to some inner product? If so, find the inner product. If not, give a proof that no such inner product exists.

**Solution:** In fact, this IS an orthogonal projection with respect to the right inner product. How do we prove this? Every orthogonal projection, $\text{proj}(v)$, has the property that $v - \text{proj}(v)$ is orthogonal to $\text{proj}(v)$. In our case, we want $\text{proj}(x, y) = (2x - y, 2x - y)$, so we need $(2x - y, 2x - y)$ to be orthogonal to $(x, y) - (2x - y, 2x - y) = (y - x, 2y - 2x)$ for all $x$ and $y$. In particular, we need $(1, 1)$ and $(1, 2)$ to be orthogonal to each other. Here is a simple way to do this: Let $S(v)$ be the linear transformation that takes $(1, 1)$ to $(1, 0)$ and $(1, 2)$ to $(0, 1)$ and define an inner product by $\langle u, v \rangle = S(u) \cdot S(v)$. That is, transform by $S$ and use the usual dot product. Can we prove this is an inner product? In fact, it is easy! I will let you check properties 1, 2, 3, and just check property 4. For that property, $\langle v, v \rangle = S(v) \cdot S(v) = ||S(v)||^2 \geq 0$. Moreover, if $\langle v, v \rangle = 0$, then $S(v) \cdot S(v) = 0$, which only happens when $S(v) = 0$. But $S$ is one-to-one so $S(v) = 0$ only when $v = 0$. 

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Finally, to get the actual formula inner product, we figure out what $S$ is. In fact, $S$ is the inverse of the transformation taking $(1, 0)$ to $(1, 1)$ and $(0, 1)$ to $(1, 2)$, so the matrix of $S$ is
\[
\begin{pmatrix}
1 & 1 \\
1 & 2
\end{pmatrix}^{-1} = \begin{pmatrix}
2 & -1 \\
-1 & 1
\end{pmatrix}.
\]
Now $S(x, y) = \begin{pmatrix}
2 & -1 \\
-1 & 1
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix}$, so in horizontal format, $S(x, y) = (2x - y, -x + y)$. This means
\[
\langle (x_1, y_1), (x_2, y_2) \rangle = S(x_1, y_1) \cdot S(x_2, y_2)
\]
\[
= (2x_1 - y_1, -x_1 + y_1) \cdot (2x_2 - y_2, -x_2 + y_2)
\]
\[
= (2x_1 - y_1)(2x_2 - y_2) + (-x_1 + y_1)(-x_2 + y_2)
\]
\[
= 5x_1x_2 - 3x_1y_2 - 3x_2y_1 + 2y_1y_2.
\]
This is just one of infinitely many possibilities. We could have let $S$ map $(1, 1), (1, 2)$ to any orthogonal basis (orthogonal with respect to the dot product) get another such formula. In particular, it is worth mentioning the following:

**Theorem** Let $A$ be any $n \times n$ invertible matrix. Then $\langle u, v \rangle = (Au) \cdot (Av)$ is an inner product on $\mathbb{R}^n$. 