From the book: 4.3.10 (Sections 4.3), 4.4.4, 4.4.6 (Sections 4.4). Also do the following:

1. One can talk about least-squares approximations in any inner product space. Given a linear operator $T$ on $V$, a least-squares solution to $T(x) = b$ is any vector $v$ for which $\|T(v) - b\|$ is a minimum.
   (a) Let $U$ be the range of $T$. Why does $T(x) = \text{proj}_U(b)$ always have to have a solution?
   (b) Prove that $v$ is a least-squares solution to $T(x) = b$ if and only if $v$ is a solution to $T(x) = \text{proj}_U(b)$.
   (c) Using the inner product $\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x) \, dx$ on $P^3$, and $D(p(x)) = p'(x)$, find a least squares solution to $T(p(x)) = x^3$.
   (d) With the same inner product as in part (c), find all least squares solutions to $D^2(p(x)) = x^3 + x^2 + x + 1$, where $D^2$ is the second derivative operator.

2. On the last homework, we used the following inner product on $P^2$.
   $$\langle p(x), q(x) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$
   (a) Using this inner product for $P^2$ and the Euclidean inner product for $R^3$, show that the linear transformation $T : P^2 \rightarrow R^3$ defined by $T(ax^2 + bx + c) = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$ is not an isometry.
   (b) Find a formula for an inner product on $R^3$ so that the $T$ is part (a) is an isometry.
   (c) Let $T : P^2 \rightarrow P^2$ be defined by $T(ax^2 + bx + c) = ax^2 + c$. Show that $T$ is a Hermitian operator with respect to this inner product.
   (d) If $D : P^2 \rightarrow P^2$ is defined by $D(p(x)) = p'(x)$, determine whether $D$ is a Hermitian operator with respect to this inner product.

For extra credit:

3. Here is a proof that if $V$ is a complex inner product space, then $\langle v, T(v) \rangle = 0$ for all $v \in V$ only when $T$ is the 0 operator.
   (a) Show that the only eigenvalue of $T$ is 0.
   (b) If $T \neq 0$, show that there is a vector $v$ in $V$ such that $T(v) \neq 0$ but $T^2(v) = 0$.
   (c) Let $v = u + T(u)$, where $T(u) \neq 0$ but $T^2(u) = 0$. Show that $\langle v, T(v) \rangle \neq 0$, and conclude the proof.
   (d) If we define $T$ on $R^2$ by $T(x, y) = (-y, x)$, show that with the Euclidean inner product on $R^2$, $\langle v, T(v) \rangle = 0$ for all $v \in R^2$. What goes wrong with the proof in parts (a), (b), (c) in this case?