

Direct Sums

Suppose that W_1 and W_2 are both subspaces of a vector space V . Recall from Chapter 2 that the sum $W_1 + W_2$ is the set $\{u + v \mid u \in W_1, v \in W_2\}$. That is, $W_1 + W_2$ is the set of all sums of vectors in W_1 with vectors in W_2 . We had the following properties of vector space sums: If W_1 and W_2 are both subspaces of V , then so are $W_1 + W_2$ and $W_1 \cap W_2$.

Theorem. If W_1 and W_2 are subspaces of V then

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Proof sketch. We go by way of bases. Let $W_1 \cap W_2$ (the smallest space) have basis $\{u_1, \dots, u_k\}$. Since $W_1 \cap W_2$ is a subspace of W_1 , this basis can be extended to a basis for W_1 : $\{u_1, \dots, u_k, v_1, \dots, v_m\}$. It may also be extended to a basis $\{u_1, \dots, u_k, w_1, \dots, w_n\}$ for W_2 . We have said that $\dim(W_1 \cap W_2) = k$, $\dim(W_1) = k + m$, $\dim(W_2) = k + n$. We must prove that

$$\dim(W_1 + W_2) = k + m + k + n - k = k + m + n.$$

An obvious candidate for a basis for $W_1 + W_2$ is the union of the three bases: $\{u_1, \dots, u_k, v_1, \dots, v_m, w_1, \dots, w_n\}$. One may check that this set is independent and spans $W_1 + W_2$ to complete the proof.

There is an important special case of the above theorem: If $W_1 \cap W_2 = \{0\}$ then $\dim(W_1 \cap W_2) = 0$ so $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$. In this case, we write $W_1 \oplus W_2$ instead of $W_1 + W_2$. We call such a sum a **direct sum**. I know of two ways to check that $V = W_1 \oplus W_2$. Method 1: Show that $W_1 \cap W_2 = \{0\}$ and $\dim(W_1) + \dim(W_2) = \dim(V)$. If so, $V = W_1 \oplus W_2$ (why?) Method 2 is to use the following result (not given in class).

Theorem. Let W_1 and W_2 be subspaces of V . Then $V = W_1 \oplus W_2$ if and only if for each vector $v \in V$, there are **unique vectors** $w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$. In words, V is a direct sum of subspaces if and only if each vector in V can be written as a sum of vectors in the subspaces in one and only one way.

Proof. " $\Rightarrow$ " If $V = W_1 \oplus W_2$, then $V = W_1 + W_2$, so every vector in V can be written as a sum of something in W_1 and something in W_2 . We must show that this sum is unique, so suppose $v = u_1 + u_2$ and $v = w_1 + w_2$, where $u_1, w_1 \in W_1$ and $u_2, w_2 \in W_2$. Then $u_1 + u_2 = w_1 + w_2$ so $(u_1 - w_1) = (w_2 - u_2)$. Letting $x = u_1 - w_1$, we have $x \in W_1$. But $x = w_2 - u_2$ also so $x \in W_2$ as well. Since $W_1 \cap W_2 = \{0\}$, it must be that $x = 0$. Thus, $u_1 - w_1 = 0 = w_2 - u_2$. Hence, $u_1 = w_1$ and $u_2 = w_2$ so the two representations of v were the same. This establishes uniqueness.

" $\Leftarrow$ " Suppose that every vector v in V can be uniquely written as a sum of something in W_1 and something in W_2 . Then by definition, $V = W_1 + W_2$. To show that $W_1 \cap W_2 = \{0\}$, let $u \in W_1 \cap W_2$. Then $u = u + 0 = 0 + u$. This gives two representations of u as a sum of something in W_1 and something in W_2 . By uniqueness, the two representations must be the same, so $0 = u$.

Invariant Subspaces

Let $T : V \rightarrow V$ be a linear operator. A subspace W of V is called a **T-invariant** subspace if for every $v \in W$, $T(v) \in W$. (In words, if v is in W , then $T(v)$ is still in W .) In such cases, we may think of T as being a linear operator on W as well as on V . **This is a different transformation** and is denoted by $T|_W$ and referred to as the restriction of T to W .

Here are two important special cases:

- (1) **Eigenspaces:** if T has c as an eigenvalue, then E_c , the eigenspace of c is an invariant space: Given $v \in E_c$, we must show that $T(v)$ is still in E_c . Since $T(v) = cv$, and E_c is a subspace, it is closed under scalar multiplication. Thus, if $v \in E_c$, then $cv = T(v) \in E_c$ as well.
- (2) **Cyclic subspaces:** Given T and any $v \in V$, the cyclic subspace of v is $W = \text{Span}\{v, T(v), T^2(v), T^3(v), \dots\}$. If V is finite dimensional, then W will be the span of finitely many of the vectors $v, T(v), T^2(v), \dots$. To show W is invariant, suppose $w \in W$. Then for some m ,

$w = c_0v + c_1T(v) + c_2T^2(v) + \cdots + c_mT^m(v)$. Now $T(w)$
 $T(w) = T(c_0v + c_1T(v) + c_2T^2(v) + \cdots + c_mT^m(v))$
 $= c_0T(v) + c_1T^2(v) + c_2T^3(v) + \cdots + c_mT^{m+1}(v),$
 so $T(w)$ is a combination of $T^k(v)$ meaning $T(w) \in W$.

Suppose that W is a T -invariant subspace of V and W has a basis $\{u_1, u_2, \dots, u_k\}$. Extend this to a basis, B , for V : $B = \{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_{n-k}\}$. We have

$$[T]_B = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B is the matrix of $T|_W$ with respect to the basis for W .

The matrix formulation of the above is as follows: Let A be an n by n matrix over a field F . A subspace W of F^n is called A -invariant if whenever any u is in W , Au is also in W . Let P be the matrix having the vectors of B as its columns. Then

$$P^{-1}AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.$$

In each case, we get a block upper triangular matrix. Since the minimal and characteristic polynomials for T are the same as the minimal and characteristic polynomials for any matrix representing T , it follows that the characteristic polynomial is the product of the characteristic polynomials of the diagonal blocks and that the minimal polynomials of the diagonal blocks each divide the minimal polynomial of T (or A).

Moreover, if V (or F^n) $= W_1 \oplus W_2$ and each of these is an invariant subspace, then $[T]$ (or $P^{-1}AP$) $= \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, a block diagonal matrix. In this case, the minimal polynomial for T (or A) is $\text{LCM}(m_B(x), m_C(x))$.

Examples:

1. Let $T: P^3 \rightarrow P^3$ be defined by $T(p(x)) = xp(x+1) - (x+1)p(x) - xp'(x)$ (as in a homework assignment.) Picking $v = x^3$, we have $T(v) = -x^3 + 3x^2 + x$, $T^2(v) = x^3 - 6x^2 + x$, $T^3(v) = \text{combination of } v, T(v), T^2(v)$. So the cyclic subspace $W = \text{Span}\{v, T(v), T^2(v), \dots\} = \text{Span}\{x^3, -x^3 + 3x^2 + x, x^3 - 6x^2 + x\}$, a 3-dimensional subspace of P^3 . If we extend this to a basis for P^3 by adjoining 1, we have $B = \{x^3, -x^3 + 3x^2 + x, x^3 - 6x^2 + x, 1\}$. Then

$$[T]_B = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which is block diagonal because $W_2 = \text{Span}\{1\}$ is also invariant and $P^3 = W \oplus W_2$. Obviously, the minimal and characteristic polynomials of the lower right hand block are both $x + 1$. For the upper left hand block, we have

$$|xI - A| = \begin{vmatrix} x & 0 & 1 \\ -1 & x & 3 \\ 0 & -1 & x+3 \end{vmatrix} = x^3 + 3x^2 + 3x + 1 = (x+1)^3.$$

By a previous homework problem, this is both the minimal and characteristic polynomial of A . Thus, the characteristic polynomial of T is $(x+1)(x+1)^3 = (x+1)^4$ and the minimal polynomial is $\text{LCM}(1+x, (1+x)^3) = (1+x)^3$.

2. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. If $v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $Av = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $A^2v = 4Av$. So the

cyclic subspace of v is $W = \{v, Av\}$, an invariant subspace of R^4 . Extending

this to a basis for R^4 , let $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Forming a matrix out of the

columns of B , let $P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Then $P^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$, and

$$\begin{aligned} P^{-1} A P &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

which is block upper triangular. The characteristic polynomial of the lower right hand block is x^2 , its minimal polynomial is 1 . The characteristic polynomial of

the upper left hand block is $\begin{vmatrix} x & 0 \\ -1 & x-4 \end{vmatrix} = x(x-4)$, which is also its minimal

polynomial. Thus, the characteristic polynomial of A is $x^3(x-4)$. Unfortunately, since $P^{-1} A P$ is not block diagonal, we cannot say that the minimal polynomial for A is $\text{LCM}(1, x(x-4)) = x(x-4)$. However, this is the case here, as one can easily check (that is, $x(x-4)$ annihilates A .)

3. Suppose that T is a transformation with eigenspace E_a . Then extending a basis for E_a to a basis B , for V , $[T]_B = \begin{pmatrix} aI & C \\ 0 & D \end{pmatrix}$. If T is diagonalizable with eigenvalues $a_1, a_2, \dots, a_k$, then $V = E_{a_1} \oplus E_{a_2} \oplus \dots \oplus E_{a_k}$, and each of these is T -invariant, so with the appropriate basis B , $[T]_B$ is block diagonal with each block being $a_i I$. That is, $[T]_B$ is diagonal, as we already knew.