Math 5330 Homework 1
Spring 2009

1. It can be shown that if a and b have no common factors, then $ax + by = n$ has integer solutions (x, y) with $x, y \geq 0$ for all sufficiently large n. Suppose that n_0 is the smallest value such that nonnegative solutions exist for $ax + by = n$, with any $n \geq n_0$. For example, $5x + 7y = n$ has a solution for all $n \geq 24$, so $n_0 = 24$.
   a. Find n_0 for the following problems:
      i. $a = 3, b = 7$ ii. $a = 4, b = 7$ iii. $a = 4, b = 11$
   b. Based on your answer to (a), using additional data if needed, guess a formula for n_0 in terms of a and b. (A proof is not needed here.)
   c. Prove that $n_0 = 24$ when $a = 5$ and $b = 7$.

2. A triangular number is a number like $10 = 1 + 2 + 3 + 4$. You can think of balls being arranged on top of each other: 4 on the bottom, then 3, then 2 and finally 1 on top, forming a triangle.
   a. If we call 0 a triangular number, find the first five positive integers that are not the sum of two triangular numbers.
   b. Can you find infinitely many numbers that are not the sum of two triangular numbers?
   c. 36 is both a square and a triangular number ($36 = 1+2+3+4+5+6+7+8$). In fact, 36 is the smallest positive integer that is both. Find the next one.

3. It is guessed that there are infinitely many twin primes. That is, pairs $n, n + 2$, where both are primes. Are there infinitely many triple primes? That is, triples of the form $n, n + 2, n + 4$ (like 3, 5, 7), where all three are primes? Explain.

Extra Credit problems:

4. Prove that the formula for n_0 that you obtained in problem 1b is correct. To do this, you must do two things: (1) Show that $n_0, n_0 + 1, \cdots$ can all be represented in the form $ax + by = n$, with $x, y \geq 0$. (2) You must show that $ax + by = n_0 - 1$ does not have a solutions with $x, y \geq 0$.

5. Show that there are infinitely many integers that are both square and triangular.