1. It can be shown that if $a$ and $b$ have no common factors, then $ax + by = n$ has integer solutions $(x, y)$ with $x, y \geq 0$ for all sufficiently large $n$. Suppose that $n_0$ is the smallest value such that nonnegative solutions exist for $ax + by = n$ for any $n \geq n_0$. For example, $5x + 7y = n$ has solutions for all $n \geq 24$ so $n_0 = 24$ when $a = 5$ and $b = 7$.

(a) Find $n_0$ for each of the following problems:
   (i) $a = 3, b = 7$  
   (ii) $a = 4, b = 7$  
   (iii) $a = 4, b = 11$

(b) Based on your answers to part a, using additional data if needed, guess a formula for $n_0$ in terms of $a$ and $b$. A proof is not needed.

(c) Prove that $n_0 = 40$ when $a = 5, b = 11$.

2. A triangular number is a number like $10 = 1 + 2 + 3 + 4$. You can think of balls being arranged on top of each other: 4 on the bottom, then 3, then 2, and finally one on top, forming a triangle.

(a) If we call 0 a triangular number, find the first five positive integers that are NOT the sum of two triangular numbers.

(b) Can you find infinitely many numbers that are not the sum of two triangular numbers?

(c) The number 36 is both a square ($36 = 6^2$) and a triangular number ($36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$). In fact, 1 is the smallest positive integer that is both a square and a triangular number, and 36 is the second. Find the next one.

3. Twin primes are pairs of numbers $n, n + 2$, where both are prime numbers. It is guessed that there are infinitely many twin primes. Are there infinitely many triple primes? That is, are there infinitely many triples $n, n + 2, n + 4$ (like 3, 5, 7) where all three are primes? Explain.

For extra credit:

4. Prove the formula for $n_0$ that you obtained in problem 1b is correct. To do this, you must do two things: (1) Show that $n_0, n_0 + 1, n_0 + 2, \ldots$ can all be represented int he form $ax + by = n$, with $x, y \geq 0$, and (2) that $ax + by = n_0 - 1$ does NOT have a solution with $x, y \geq 0$.

5. Show that there are infinitely many integers that are both squares and triangular numbers.

6. Show that Conjecture 1 in the introduction is true when $k = 2$. That is, show that whenever
   \[
   \frac{(1 - x^n)(1 - x^{n+1})}{(1 - x^m)(1 - x^{m+1})}
   \]
   is a polynomial, it has nonnegative coefficients. What about $k = 3$?