Math 5330 Homework 2
Spring 2009

1. Find a description of all Pythagorean triples of the for $(x, y, y + 1)$.
(You don't need the formula for Pythagorean triples to do this problem.)

2. Use the fact that every Pythagorean triple (a, b, c) where b is even is of the form $(d(x^2 - y^2), 2dxy, d(x^2 + y^2))$ to:
   a. Prove that there are only finitely many Pythagorean triples using 52. (That is, there are only finitely many (a, b, c) with $a^2 + b^2 + c^2$, where one of a, b, c is 52.)
   b. Find all Pythagorean triples using 52.

3. Find all solutions to $x^2 + 2y^2 = z^2$ in positive integers, where x, y, z are all relatively prime, using the geometric approach.

4. For all its problems, the method of introducing "new" integers has one good property: it allows one to find infinitely many solutions to some equations quite easily. It just doesn't guarantee that all solutions have that form.
   a. Use this method to show that the equation $x^2 + ny^2 = z^2$ always has infinitely many solutions. (Show, in fact, that there are always solutions of the form. $(x, y, z) = (|p^2 - nq^2|, 2pq, p^2 + nq^2)$.)
   b. Use (a) to find a primitive solution (x, y, z) to $x^2 + 10y^2 = z^2$.
   c. Find a primitive solution (x, y, z) to $x^2 + 10y^2 = z^2$ that isn't of the form given in (a).

5. Find an infinite number of solutions to each of the following equations. Note: I am not asking you to find all solutions, just an infinite number.
   a. $x^3 + y^3 = z^2$
   b. $x^2 + y^2 = z^3$ Hint: try $x = n(n^2 - 3)$, $y = 3n^2 - 1$.

Extra Credit problems:

1. Find infinitely many primitive solutions to $x^3 + y^3 = z^2$.

2. Find all primitive solutions to $x^2 + y^2 = z^3$. Hint: Use the fact that the Gaussian integers have unique factorization.