Math 5330 Homework 3
Spring 2009

1. Prove that $\sqrt{3}$ is irrational
   a. Using the Fundamental Theorem of Arithmetic.
   b. Using Infinite Descent.

2. Prove that $\sqrt{5}$ is irrational
   a. Using the Fundamental Theorem of Arithmetic.
   b. Using Infinite Descent.

3. Prove that $\sqrt[3]{2}$ is irrational.
   a. Using the Fundamental Theorem.
   b. Using Infinite Descent. Hint: Suppose that $\sqrt[3]{2}$ is rational. Then there must be an integer n such that $n\sqrt[3]{2}$ and $n\sqrt[3]{4}$ are both integers. (Why?) Now use an argument by infinite descent on n.

4. Prove that $\log_{10} 2$ is irrational. Hint: if $\log_{10} 2$ were rational, then there would be positive integers m and n such that $\log_{10} 2 = \frac{m}{n}$. Clear fractions, exponentiate and use unique factorization to get a contradiction.

5. In general, it is true that $ax^2 + by^2 = cz^2$ has either infinitely many primitive solutions or no solutions other than (0, 0, 0).
   a. Show that $x^2 + y^2 = 5z^2$ has infinitely many primitive solutions.
   b. Show that $x^2 + y^2 = 6z^2$ has no solutions other than (0, 0, 0).
   c. To which category does $x^2 + y^2 = 7z^2$ belong?

6. Let E be the set of positive numbers: $E = \{2, 4, 6, 8, \cdots\}$. We can define divisibility in E: we say that n is divisible by k if for some number q **in E**, n = kq. For example, 40 is divisible by 10 since 40 = 4*10, but 30 is not divisible by 10 in E. We say that p is a prime in E if p has no divisors. For example, 2 and 30 are examples of primes in E.
   a. Find the primes in E with $p \leq 50$.
   b. Give an example to show that E does not have unique factorization.

Extra Credit problems:

1. Prove the statement in problem 5, that $x^2 + y^2 = kz^2$ has either no solutions or infinitely many primitive solutions. Hint: show that if there is even one solution, there has to be infinitely many primitive ones.

2. Find all primitive solutions to $x^2 + y^2 = z^3$. Hint: Use the fact that the Gaussian integers have unique factorization.

3. We mentioned in class that $\left\{a + b\sqrt{-7} \mid a, b \text{ are integers}\right\}$ does not have unique factorization. This was the reason that we missed solutions when we tried to solve $x^2 + 7y^2 = z^2$ in class. (In particular, we missed the solution $x = 3$, $y = 1$, $z = 4$.) If we let $\omega = \frac{-1 + \sqrt{-7}}{2}$, then it turns out that $F = (a + b\omega \mid a, b \text{ are integers}\}$ DOES have unique factorization. Use this to find a description of all primitive solutions to $x^2 + 7y^2 = z^2$.

4. Find all primitive solutions to $x^3 + y^3 = z^2$. Hint: use ideas in E1, E2.

5. Say as much as you can about those k for which $x^2 + y^2 = kz^2$ has infinitely many primitive solutions.