1. It can be shown that if \( a \) and \( b \) have no common factors, then \( ax + by = n \) has integer solutions \((x, y)\) with \( x, y \geq 0 \) for all sufficiently large \( n \). Suppose that \( n_0 \) is the smallest value such that nonnegative solutions exist for \( ax + by = n \) for any \( n \geq n_0 \). For example, \( 5x + 7y = n \) has solutions for all \( n \geq 24 \) so \( n_0 = 24 \) when \( a = 5 \) and \( b = 7 \).

(a) Find \( n_0 \) for each of the following problems:

(i) \( a = 3, b = 7 \)  
(ii) \( a = 4, b = 7 \)  
(iii) \( a = 4, b = 11 \)

Solution: Here are the numbers representable in each of the cases:

(i) \( 0, 3, 6, 7, 9, 10, 12, 13, 14, 15, 16, \ldots \), with \( n_0 = 12 \),  
(ii) \( 0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 23, \ldots \), and \( n_0 = 18 \),  
(iii) \( 0, 4, 8, 11, 12, 15, 16, 19, 20, 22, 23, 24, 26, 27, 28, 30, 31, 32, 33, 34, \ldots \) giving \( n_0 = 30 \).

(b) Based on your answers to part a, using additional data if needed, guess a formula for \( n_0 \) in terms of \( a \) and \( b \). A proof is not needed.

Solution: The formula is \( n_0 = (a - 1)(b - 1) \).

Checking pairs like \((2, 7), (3, 7), (4, 7), (5, 7)\) shows that as \( a \) increases, \( n_0 \) increases by \( 6 = b - 1 \) each time. One could collect more data to verify this pattern. If \( n_0 \) is a multiple of \( b - 1 \), then symmetry would demand that it also be a multiple of \( a - 1 \), and \((a - 1)(b - 1)\) is the simplest such formula. Moreover, it works in all the cases investigated.

(c) Prove that \( n_0 = 40 \) when \( a = 5, b = 11 \).

Solution: We first check that \( 5x + 11y = 39 \) has no nonnegative integer solutions. One way to do this: if \( 5x + 11y = 39 \) then \( 5x = 39 - 11y \), so we subtract multiples of 11. Since none of 39, 28, 17, or 6 are multiples of 5, there are no solutions. This shows that \( n_0 \geq 40 \). Since \( 40 = 5 \cdot 8 + 11 \cdot 0, 41 = 5 \cdot 6 + 11 \cdot 1, 42 = 5 \cdot 4 + 11 \cdot 2, 43 = 5 \cdot 2 + 11 \cdot 3, \) and \( 44 = 5 \cdot 0 + 11 \cdot 4 \), we have five things representable in a row. Once we have these, we can add multiples of 5 to get everything thereafter. For example, if we want 83, we use 43 and add as many multiples of 5 as we need (8 of them) to get from 43 to 83. A more formal proof would look like this: we know we can represent 40, 41, 42, 43, 44. Suppose we can represent 40, 41, 42, \ldots, \( k \) for some \( k \geq 44 \). Then we can also represent \( k + 1 \) by adding 5 to the right thing. In this case, we add 5 to the thing five back from \( k + 1 \). That is, since \( k \geq 44, (k + 1) - 5 = k - 4 \geq 40 \). This is in the range of things we can represent, so \( k - 4 = 5x + 11y \) for some \( x, y \). Now \( k + 1 = 5(x + 1) + 11y \) so we can represent \( k + 1 \) if we can represent everything from 40 to \( k \).
2. A triangular number is a number like 10 = 1 + 2 + 3 + 4 or 15 = 1 + 2 + 3 + 4 + 5. More generally, \( T_n = 1 + 2 + \cdots + n \).

(a) Find the first five positive integers that are NOT the sum of two triangular numbers.

Solution: Here are the first 10 triangular numbers: 0, 1, 3, 6, 10, 15, 21, 28, 36, 45. We add 1 to everything on this list, then 3, then 6, and so on. The list of numbers representable as the sum of two triangular numbers is 0, 1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 13, 15, 16, 18, 20, 21, 22, 24, 25, 27, 28, 29, . . . . Numbers that are not the sums of triangular numbers are 5, 8, 14, 17, 19, 23, 26, . . . . The first five are 5, 8, 14, 17, 19.

(b) Can you find infinitely many numbers that are not the sum of two triangular numbers?

Solution: In fact, there are infinitely many. If \( m \) is any number that leaves a remainder of 5 when divided by 9, then \( m \) is not the sum of two triangular numbers. That is, no number of the form 9\( k \)+5 is the sum of two triangular numbers.

To verify this, let \( T_n \) be the \( n \)'th triangular number. Then \( T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \). Now let \( n = 9k + r \). We have

\[
T_n = \frac{(9k + r)(9k + r + 1)}{2} = \frac{9k(9k + r + 1) + 9kr}{2} + \frac{r(r + 1)}{2}
\]

I will leave it to you to check that \( \frac{9k(9k + r + 1) + 9kr}{2} \) is an integer which is divisible by 9. Consequently, \( T_n \) and \( T_r \) leave the same remainder when divided by 9. This means possible remainders are the remainders for the first 9 triangular numbers, 0, 1, 3, 6, 1, 6, 3, 1, 0, or just 0, 1, 3, 6. When we add two triangular numbers together, we can get any sum of these two (when reduced by 9 if needed,) giving the list 0, 1, 2, 3, 4, 6, 7. That is, when the sum of two triangular numbers is divided by 9, only these remainders can occur. In particular, 5 and 8 cannot occur as remainders, so no number of the form 9\( k \)+5 (or 9\( k \)+8) can be written as the sum of two triangular numbers.

Solution: A second approach, from the class goes like this: Let’s see if we can estimate how many numbers up to \( T_n = \frac{n(n+1)}{2} \) are the sum of two triangular numbers. We can only use \( T_k < T_n \) so we get a total of \( n \) values of \( k \) (those \( k \) with 0 \( \leq k \) \( \leq n - 1 \)). For example, \( T_7 = 28 \), so suppose we want all sums of triangular numbers up to 27. We can only use \( T_0, \ldots, T_6 \) in these sums.
If we have \( n \) triangular numbers to work with, how many sums can we get? Since \( T_i + T_j = T_j + T_i \), we can get at most \( \frac{n(n+1)}{2} = T_n \) sums (\( n \) cases when \( i = j \) and \( T_{n-1} \) cases when \( i > j \)). This means that there are \( T_n \) nonnegative integers less than \( T_n \) and exactly \( T_n \) sums of two triangular numbers. But some of these sums are too big! In our example above, \( T_5 + T_6 = 15 + 21 > 27 \). Let’s just look for easy cases: When is \( T_k + T_k = 2T_k > T_n \)? This will happen when \( k^2 + k > \frac{n(n+1)}{2} \). Multiplying by 4 and adding 1, we need \((2k+1)^2 = 4k^2 + 4k + 1 > 2n(n+1)+1\). It turns out that \( k \geq \frac{3}{4}n \) is good enough to make this true so there are \( n - \frac{3}{4}n = \frac{1}{4}n \) values of \( k \) where the sum is too large. This means we miss at least \( \frac{1}{4}n \) of the values up to \( T_n \), and since this grows as \( n \) grows, infinitely many numbers are not the sum of two triangular numbers.

(c) The number 36 is both a square \((36 = 6^2)\) and a triangular number \((36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8)\). In fact, 1 is the smallest positive integer that is both a square and a triangular number, and 36 is the second. Find the next one.

**Solution:** The next one is 1225: \( 1225 = 35^2 \) and \( 1225 = 1 + 2 + 3 + \cdots + 49 \).

3. Twin primes are pairs of numbers \( n, n + 2 \), where both are prime numbers. It is guessed that there are infinitely many twin primes. Are there infinitely many triple primes? That is, are there infinitely many triples \( n, n + 2, n + 4 \) (like 3, 5, 7) where all three are primes? Explain.

**Solution:** No, \((3, 5, 7)\) is the only one. The reason: one of \( n, n + 2, n + 4 \) is always divisible by 3, and there is only one prime divisible by 3, namely 3. Why is one of \( n, n + 2, n + 4 \) divisible by 3? We can see this with cases: Every number \( n \) can be written \( n = 3k + r \), where \( r \) is 0, 1, or 2. If \( r = 0 \), then \( n \) is divisible by 3, if \( r = 1 \), then \( n + 2 \) is divisible by 3, and if \( r = 2 \), then \( n + 4 \) is divisible by 3.
4. Prove the formula for $n_0$ that you obtained in problem 1b is correct. To do this, you must do two things: (1) Show that $n_0, n_0 + 1, n_0 + 2, \ldots$ can all be represented in the form $ax + by = n$, with $x, y \geq 0$, and (2) that $ax + by = n_0 - 1$ does NOT have a solution with $x, y \geq 0$.

**Solution:** This is only a partial solution. We can show that $(a - b)(b - 1) - 1$ is not the sum of $a$'s and $b$'s. To do this, suppose that $(a - 1)(b - 1) - 1 = ab - a - b = ax + by$. Rewrite this $ab - a - ax = by + b$, or $a(b - 1 - x) = b(y + 1)$. This tells us that $a$ is a divisor of $b(y + 1)$. But $a$ and $b$ are relatively prime so $a$ has to be a divisor of $y + 1$. Since $y + 1 \geq 1$ it must be that $y \geq a - 1$. By symmetry, $x \geq b - 1$. Consequently, $ax + by \geq a(b - 1) + b(a - 1) = 2ab - a - b > ab - a - b$, a contradiction.

Showing that every number $n \geq (a - 1)(b - 1)$ is the sum of $a$'s and $b$'s is harder. Here is one approach, with $a = 4, b = 11$ as an example. First, we may assume that $1 < a < b$ because if $a = 1$, all numbers are combinations of $a$ and $b$, and by symmetry, we can take $a$ to be the smaller of the two numbers. Next, for any $n$ we can divide by $a$ and write $n = qa + r$, where $0 \leq r \leq a - 1$. The numbers $0, a, 2a, \ldots$ all have $r = 0$ and can be represented. Similarly, the numbers $b + a, b + 2a, \ldots$ all have the same $r$, and can all be represented. In fact, it is not hard to check that $k_1b$ and $k_2b$ have different remainders $r$ as long as $0 \leq k_1 < k_2 \leq a - 1$. In the case of $a = 4, b = 11$, the remainders are 0, 3, 2, 1 for 0, 11, 22, 33.

Now among any progression $r, r + a, r + 2a, \ldots$, the first time something in the progression has the form $ax + by$ will be something $kb$. For example, what is the first number $n = 4x + 11y$ with a remainder of 1 when divided by 4? All multiples of 4 have remainder 0, $11 +$ multiples of 4 have remainder 3, $22 +$ multiples of 4 have remainder 2, and finally, 33 numbers of 4 have remainder 1. So 33 is the smallest number $n = 4x + 11y$ with a remainder of 1. Since all remainders have been covered by the time we get to $(a - 1)b$, all $n \geq (a - 1)b$ can be written in terms of $a$'s and $b$'s. Moreover, $(a - 1)b - a$ has the same remainder as $(a - 1)b$ so it can't be written that way. Now $(a - 1)b - a = (a - 1)(b - 1) - 1$, which again shows that $n_0 \geq (a - 1)(b - 1)$. Suppose that $r$ is the remainder of $(a - 1)b$. This is the last remainder to be covered, and it is covered for the first time at $(a - 1)b$. The second to last remainder to be covered is the remainder for $(a - 2)b$, and every remainder other than $r$ will have been covered by this time. But $(a - 2)b = (a - 1)b - b > (a - 1)b - a = (a - 1)(b - 1) - 1$, so every remainder other than $r$ has been covered by this time. In particular, the remainders for $(a - 1)(b - 1), (a - 1)(b - 1) + 1, \ldots, (a - 1)(b - 1) + (a - 2)$ have all been covered, so these $a - 1$ numbers are representable, as is the next one, $(a - 1)(b - 1) + a - 1 = (a - 1)b$. This gives us $a$ numbers in a row, so everything from $(a - 1)(b - 1)$ on is the sum of $a$'s and $b$'s.
5. Show that there are infinitely many integers that are both squares and triangular numbers.

**Solution:** One way: If \( \frac{n(n+1)}{2} = m^2 \) then multiplying by 8 gives \( 4n^2 + 4n = 8m^2 \), or \( 4n^2 + 4n + 1 = 8m^2 + 1 \), which we write \( (2n+1)^2 = 8m^2 + 1 \), or \( (2n+1)^2 - 8m^2 = 1 \). An equation of this form is called a Pell equation. If we think of it as \( x^2 - 8y^2 = 1 \), then write this \( (x + \sqrt{8}y)(x - \sqrt{8}y) = 1 \). Raising this to the \( j \)'th power, \( (x + \sqrt{8}y)^j(x - \sqrt{8}y)^j = 1 \). If we were to expand \( (x + \sqrt{8})^j \) out, writing it \( (x + \sqrt{8})^j = k + \sqrt{8}m \), then \( (x - \sqrt{8})^j = k - \sqrt{8}m \), and taking the product, \( 1 = (k + \sqrt{8}m)(k - \sqrt{8}m) = k^2 - 8m^2 \). If \( k \) is also odd, with \( k = 2n+1 \) then we get a triangular number equal to a square.

So can we find a solution to \( x^2 - 8y^2 = 1 \)? Yes: \( x = 3, y = 1 \). Squaring \( 3 + \sqrt{8} \) we get

\[
(3 + \sqrt{8})^2 = 9 + 6\sqrt{8} + 8 = 17 + 6\sqrt{8}.
\]

17 is odd so \( 2n+1 = 17 \rightarrow n = 8 \), and \( \frac{8 \cdot 9}{2} = 36 \). Cubing \( 3 + \sqrt{8} \),

\[
(3 + \sqrt{8})^3 = (17 + 6\sqrt{8})(3 + \sqrt{8}) = 51 + 17\sqrt{8} + 18\sqrt{8} + 48 = 99 + 35\sqrt{8}.
\]

Again, 99 is odd so \( 2n+1 = 99 \rightarrow n = 49 \), and \( \frac{49 \cdot 50}{2} = 1225 \). In fact, all powers of \( 3 + \sqrt{8} \) will have odd values for \( k \) so we get infinitely many solutions. The solutions grow exponentially fast, however.

**Solution:** Here is another solution from the class that is very nice. Suppose that \( \frac{n(n+1)}{2} = m^2 \) for some \( m \) and \( n \). Then we can find larger values that also work by replacing \( n \) by \( 4n^2 + 4n \)!

That is, \[
\frac{(4n^2 + 4n)(4n^2 + 4n + 1)}{2} = \frac{4n(n+1)(2n+1)^2}{2} = \frac{n(n+1)}{2}4(2n+1)^2 = m^24(2n+1)^2 = (2m(2n+1))^2.
\]

That is, if \( T_n \) is a perfect square, then so is \( T_{4n^2+4n} \).
6. Show that Conjecture 1 in the introduction is true when $k = 2$. That is, show that whenever \( \frac{(1 - x^m)(1 - x^{n+1})}{(1 - x^n)(1 - x^{m+1})} \) is a polynomial, it has nonnegative coefficients. What about $k = 3$?

**Solution:** From the introductory notes, we said: \( \frac{(1 - x^m)(1 - x^n)}{(1 - x^a)(1 - x^b)} \) is a polynomial with nonnegative coefficients exactly when one of the following two conditions is met.

1. $m$ is divisible by $a$ or $b$ and $n$ is divisible by the other, or

2. one of the numbers $m, n$, say $m$ is divisible by both $a$ and $b$ and there is a nonnegative integer solution to $ax + by = n$.

Also, \( \frac{(1 - x^m)(1 - x^n)}{(1 - x^a)(1 - x^b)} \) is a polynomial provided $m$ is divisible by $a$ or $b$, and $n$ is divisible by the other, or one of $m, n$ is divisible by both $a$ and $b$. I will let you see if you can prove this.

With these in place, if $n$ is divisible by $m$ and $n + 1$ is divisible by $m + 1$, then the result is a polynomial with positive coefficients. Similarly, if $n$ is divisible by $m + 1$ and $n + 1$ is divisible by $m$, the expression is a polynomial with nonnegative coefficients. This leaves two other cases: $n$ is divisible by both $m$ and $m + 1$ or $n + 1$ is divisible by both $m$ and $m + 1$. In this case, the expression will be a polynomial, but by the theorem, it will have nonnegative coefficients only if the second numerator term is a nonnegative linear combination of the denominator terms. Now $m$ and $m + 1$ are relatively prime so by the formula we guessed in problem 1, every number from $(m - 1)m$ on is a combination of $m$ and $m + 1$. If, say, $n + 1$ is divisible by both $m$ and $m + 1$ then $n + 1 \geq m(m + 1) = m(m - 1) + 2m$ so $n \geq m(m - 1) + 2m - 1 \geq m(m - 1)$. As a consequence, $n$ is a nonnegative combination of $m$ and $m + 1$ so the polynomial has nonnegative coefficients.

The case with $k = 3$ is considerably harder.