1. Find a description of all Pythagorean triples of the form \((x, y, y+1)\). You do NOT need the formula for Pythagorean triples to do this problem.

**Solution:** We let \(z = y + 1\): \(x^2 + y^2 = (y + 1)^2 = y^2 + 2y + 1\), so \(x^2 = 2y + 1\). If we solve for \(y\) we get \(y = \frac{x^2 - 1}{2}\). In order for this to be an integer, we need \(x\) to be odd so let \(x = 2k + 1\). Then \(y = \frac{4k^2 + 4k}{2} = 2k^2 + 2k\). All triples will have the form \((2k + 1, 2k^2 + 2k; 2k^2 + 2k + 1)\). For example, when \(k = 4\), we get \((9, 40, 41)\).

2. Use the fact that every Pythagorean triple \((a, b, c)\) has one of the two forms

\[(d(p^2 - q^2), 2dpq, d(p^2 + q^2)) \text{ or } (2dpq, d(p^2 - q^2), d(p^2 + q^2))\]

to find all Pythagorean triples containing 68. That is, any of \(x, y, z\) is allowed to be 68.

**Solution:** The easiest way to do this is to make use of the properties of \(p\) and \(q\). That is, we need \(0 < q < p\), \(p\) and \(q\) are relatively prime, and one of them is even. For \(x = 68\), we need \(d(p^2 - q^2) = 68\), and \(p^2 - q^2\) should be odd. This means that \(d\) needs to be divisible by 4, so \(d = 4\) or \(d = 68\). We can’t have \(d = 68\) because then \(p^2 - q^2 = 1\), and this does not have solutions with \(0 < q < p\). So \(d = 4\), \(p^2 - q^2 = 17\).

To find \(p\) and \(q\), write \((p + q)(p - q) = 17\). We need \(p + q = 17\), \(p - q = 1\), so \(p = 9\), \(q = 8\), \(d = 4\), giving \((68, 576, 580)\). Next, consider \(y = 68\). We have \(2dpq = 68\), or \(dpq = 34\). Moreover, we need \(0 < q < p\) and one of \(p\) and \(q\) is even. The cases:

\(p = 34, q = 1, d = 1, p = 17, q = 2, d = 1, p = 2, q = 1, d = 17\). These lead to the triples \((1155, 68, 1157), (285, 68, 293), (51, 68, 85)\). Finally, suppose that \(z = 68\), so \(68 = d(p^2 + q^2)\). Again, \(p^2 + q^2\) is odd so \(d\) must be a multiple of 4. Since 4 and 68 are the only divisors of 68 which are divisible by 4, we need \(d = 4\) or \(d = 68\). Since \(p^2 + q^2 \geq 2^2 + 1^2 = 5\) the only possibility is \(d = 4\). We have \(p^2 + q^2 = 17\), or \(p = 4, q = 1, d = 4\). This gives the triple \((60, 32, 68)\).

We got the five triples \((68, 576, 580), (1155, 68, 1157), (285, 68, 293), (51, 68, 85)\) and \((60, 32, 68)\). If we interchange \(x\) and \(y\) we get five more: \((576, 68, 580), (68, 1155, 1157), (68, 285, 293), (68, 51, 85)\) and \((32, 60, 68)\), for a total of 10 triples.

3. Use the geometric approach to find all solutions to \(x^2 + 2y^2 = z^2\).

**Solution:** As I mentioned in class, we already know the answer: the primitive triples should all have the form \((x, y, z) = (|p^2 - 2q^2|, 2pq, p^2 + 2q^2)\), where \(p\) and \(q\) are relatively prime and \(p\) is odd. How do we get this (in particular, the absolute value) geometrically? We parameterize the curve \(x^2 + 2y^2 = 1\) the same way we did with
the circle. This curve passes through (-1, 0), so we look for all lines through that point that intersect the curve in the first quadrant. This lines all have the form \( y = m(x+1) \) again. We have \( x^2 + 2(m(x+1))^2 = 1 \). We know one solution: \( x = -1 \), we want the other. A trick: we write this \( x^2 - 1 + 2m^2(x+1)^2 = 0 \), and use the fact that \( x^2 - 1 = (x-1)(x+1) \). This means \( x + 1 \) is a factor of both terms, and if we divide by \( x + 1 \) we get \( x - 1 + 2m^2(x+1) = 0 \), so \( (1+2m^2)x + 2m^2 - 1 = 0 \). Thus, we get \( x = \frac{1 - 2m^2}{1 + 2m^2} \), and \( y = m(x+1) = \frac{2m}{1 + 2m^2} \). To be in the first quadrant, we need \( 0 < m < \frac{1}{\sqrt{2}} \).

To get triples, we write \( \left( x' = \frac{x}{z}, y' = \frac{y}{z} \right) = \left( \frac{1 - 2m^2}{1 + 2m^2}, \frac{2m}{1 + 2m^2} \right) \). To avoid too much confusion, I will replace \( m \) by \( \frac{b}{a} \) where \( a \) and \( b \) are relatively prime and \( 0 < b < \frac{1}{\sqrt{2}}a \).

Some simplification gives \( \left( x' = \frac{a^2 - 2b^2}{a^2 + 2b^2}, \frac{2ab}{a^2 + 2b^2} \right) \), and we try to get back to \( x, y, \) and \( z \). If \( a^2 - 2b^2 \) is relatively prime to \( a^2 + 2b^2 \), then we get \( (x, y, z) = (a^2 - 2b^2, 2ab, a^2 + 2b^2) \), which is close to the answer we want. Given that \( a \) and \( b \) are relatively prime, what factors can \( a^2 - 2b^2 \) and \( a^2 + 2b^2 \) have in common? As usual, if \( d \) is a common divisor, then \( d \) divides their sum, \( 2a^2 \) and their difference, \( 4b^2 \). Since \( a \) and \( b \) are relatively prime, \( d \) must be a divisor of 2, so \( d \) is either 1 or 2.

We’ve done the case \( d = 1 \), when can \( d = 2 \)? We need \( a^2 + 2b^2 \) to be divisible by 2, and since \( 2b^2 \) already is, this means \( a \) must be even. Writing \( a = 2c \) we have

\[
\frac{a^2 - 2b^2}{a^2 + 2b^2} = \frac{(2c)^2 - 2b^2}{(2c)^2 + 2b^2} = \frac{2c^2 - b^2}{2c^2 + b^2},
\]

and \( (x, y, z) = (2c^2 - b^2, 2bc, b^2 + 2c^2) \). In this case, since \( a \) and \( b \) are relatively prime, \( b \) must be odd. Thus, we have two cases: If \( a \) is odd, \( (x, y, z) = (a^2 - 2b^2, 2ab, a^2 + 2b^2) \), and if we replace \( a \) by \( p \) and \( b \) by \( q \) we get \( (p^2 - 2q^2, 2pq, p^2 + 2q^2) \). If \( a \) is even and \( b \) is odd, then \( (x, y, z) = (2c^2 - b^2, 2bc, b^2 + 2c^2) \) and if we let \( b = p, c = q \) we get \( (2q^2 - p^2, 2pq, p^2 + 2q^2) \). In each case, \( p \) is odd. We can combine the two cases using the absolute value:

All primitive solutions to \( x^2 + 2y^2 = z^2 \) have the form \((|p^2 - 2q^2|, 2pq, p^2 + 2q^2)\), where \( p \) and \( q \) are relatively prime and \( p \) is odd. If we want all solutions, we introduce an integer, \( d \), and get \( (x, y, z) = (d|p^2 - 2q^2|, 2dpq, d(p^2 + 2q^2)) \).
4. This question considers solutions to the equation $x^2 + ny^2 = z^2$.

(a) Verify that this equation has the infinite family of solutions $(|p^2 - nq^2|, 2pq, p^2 + nq^2)$.

**Solution:** This is just an algebra exercise:

\[ x^2 + ny^2 = |p^2 - nq^2|^2 + n(2pq)^2 = p^4 - 2np^2q^2 + n^2q^4 + 4np^2q^2 = p^4 + 2np^2q^2 + n^2q^4 = (p^2 + nq^2)^2 = z^2. \]

(b) Use part (a) to find three primitive solutions to $x^2 + 10y^2 = z^2$.

**Solution:** We need only pick appropriate values of $p$ and $q$. If we pick $p$ and $q$ to be relatively prime, all we have to avoid is even $p$ or $p$ divisible by 5. So letting $p = q = 1$ gives $x = |1 - 10|$, $y = 2 \cdot 1 \cdot 1$, $z = 1 + 10$, leading to the triple $(9, 2, 11)$. Letting $p = 3$, $q = 1$ gives $(1, 6, 19)$ and $p = 7$, $q = 2$ gives $(9, 28, 89)$.

(c) Find a primitive solution to $x^2 + 10y^2 = z^2$ that is not of the form given in part a.

**Solution:** One example is $(3, 2, 7)$. If $2pq = 2$ then $p = q = 1$, but that gives $(9, 2, 11)$ not $(3, 2, 7)$.

5. Find an infinite number of solutions to each of the following equations. Note: I am not asking you to find all solutions, or even infinitely many primitive solutions, just an infinite number.

(a) $x^3 + y^3 = z^2$

**Solution:** Since I did not ask for primitive solutions, one way to do this problem is to find one solution and modify it to get other solutions. The most common ways to do this: If $x = y = 2^k$ for some $k$ then $z^2 = 2(2^k)^3 = 2^{3k+1}$. If we pick $k$ to be odd, say $k = 2n + 1$, then $x = y = 2^{2n+1}$ and $z = 2^{3n+2}$, so we can get infinitely many triples of the form $(2^{2n+1}, 2^{2n+1}, 2^{3n+2})$.

Alternatively, find one solution, say $x = 1$, $y = 2$, $z = 3$, and then modify by some integer $d$. We cannot just multiply by $d$ because we no longer have a solution. However, we can multiply $x$ and $y$ by $d^2$ and $z$ by $d^3$. Using the initial triple $(1, 2, 3)$, we get the family $(d^2, 2d^2, 3d^3)$.

(b) $x^2 + y^2 = z^3$

**Solution:** We can again use either of the techniques used in part a. The first approach would lead to triples $(2^{3n+1}, 2^{3n+1}, 2^{2n+1})$. On the other hand, if we were to start, say, with triple $(2, 11, 5)$, we could get $(2d^3, 11d^5, 5d^2)$. 

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For extra credit:

6. It turns out that $x^2 + 10y^2 = z^2$ has infinitely many (positive integer) solutions with $x = 1$. Can you find 5 of them?

**Solution:** The values of $y$ and $z$ grow quickly. I, at least, only found 3 with a computer search. However, setting $x = 1$, we want $z^2 − 10y^2 = 1$, or $1 = (z − \sqrt{10}y)(z + \sqrt{10}y)$. If we can find one solution, we will have infinitely many by raising this equation to the $n$’th power for $n = 2, 3, \ldots$. The smallest solution I found in my computer search was $x = 1, y = 6, z = 19$. To get other solutions, write $z + y\sqrt{10} = (19 + 6\sqrt{10})^n$, for $n = 2, 3, 4, 5$. This leads to four more solutions:

$$(1, 228, 721), (1, 8658, 27379), (1, 328776, 1039981), (1, 12484830, 39480499).$$

7. Chicken McNuggets used to come in denominations of 6 pieces, 9 pieces, and 20 pieces. The Chicken McNugget problem: What is the largest number of Chicken McNuggets one cannot get by buying some combination of the above? That is, what is the largest positive integer, $n$, for which $6x + 9y + 20z = n$ has no solution in nonnegative integers $x, y, z$? Prove your answer correct.

**Solution:** The answer is $n = 43$. To prove this, we need two things: We CANNOT write 43 = $6x + 9y + 20z$ for any nonnegative integers $x, y, z$, but we CAN do so for any integer greater than 43. For the second part, all we need do is get six in a row since we could add 6’s to any of those six numbers to get any larger numbers. So we have $44 = 4 \times 6 + 20$, $45 = 5 \times 9$, $46 = 6 + 2 \times 20$, $47 = 3 \times 9 + 20$, $48 = 8 \times 6$, $49 = 9 + 2 \times 20$. To show that 43 is not representable, note that using 6’s and 9’s, you can only get multiples of 3. Thus, to represent 43, you would need to use 20’s. Using one 20 will not work because 23 is also not divisible by 3. Using 2 20’s will not work because even though 3 is divisible by 3, you can’t get it with 6’s and 9’s.

8. Find infinitely many primitive solutions to the equations in problem 5.

**Solution:** One of these is not to hard: for $x^2 + y^2 = z^3$, use Gaussian integers. We write $z^3 = (x + iyz)(x − iy)$ and ask that each of $x + iy$ and $x − iy$ be cubes (rather than squares as when we had $z^2$. If we write $x + iy = (p + qi)^3$, then expanding gives $x + iy = p^3 + 3p^2qi − 3pq^2 − q^3i$, so $x = p^3 − 3pq^2, y = 3p^2q − q^3$. A calculation shows that $z = p^2 + q^2$. For infinitely many solutions we don’t need both $p$ and $q$ to be free. We could set $q = 1$ so $x = p^3 − 3p, y = 3p^2 − 1, z = p^2 + 1$. This will be primitive whenever $p$ is even. For example, $p = 2 \rightarrow x = 2, y = 11, z = 5$, or
The equation \( x^3 + y^3 = z^2 \) is much harder. The problem is that \( x^3 + y^3 \) has a messier factorization: \( x^3 + y^3 = (x + y)(x^2 - xy + y^2) \). I made use of problem (4) with \( n = 3 \) to help on this: An infinite family of solutions to \( x^2 + 3y^2 = z^2 \) is \((p^2 - 3q^2), 2pq, p^2 + 3q^2 \). To make use of this, I first asked that \( y \) be even, and replacing \( y \) by \( 2y \), I looked at \( x^3 + 8y^3 = z^2 \). The algebra is simpler here. In this case, \( x^3 + 8y^3 = (x + 2y)(x^2 - 2xy + 4y^2) \). I looked for solutions in which both terms are squares. That is, I want \( x + 2y = a^2 \), \( x^2 - 2xy + 4y^2 = b^2 \).

Start with the second equation. We can rewrite it \( (x - y)^2 + 3y^2 = b^2 \). By problem (4), we can get a solutions using

\[
x - y = c^2 - 3d^2, \quad y = 2cd, \quad b = c^2 + 3d^2.
\]

This means \( x = c^2 + 2cd - 3d^2 \) and since \( x + 2y = a^2 \), we have \( c^2 + 6cd - 3d^2 = a^2 \). Now work with this equation. Completing the square, \( (c + 3d)^2 - 12d^2 = a^2 \), which we can rewrite \( a^2 + 3(2d)^2 = (c + 3d)^2 \). We again use problem 4 to say

\[
a = p^2 - 3q^2, \quad 2d = 2pq, \quad c + 3d = p^2 + 3q^2.
\]

Now we try to get back to \( x, y, z \). Using \( d = pq \) we get \( c = p^2 - 3pq + 3q^2 \). We have \( y = 2cd = 2pq(p^2 - 3pq + 3q^2) \) and \( x = c^2 + 2cd - 3d^2 = p^4 - 4p^3q + 6p^2q^2 - 12pq^3 + 9q^4 \). A calculation gives \( z = (p^2 - 3q^2)(p^4 - 6p^3q + 18p^2q^2 - 18pq^3 + 9q^4) \). The triple is \( (x, 2y, z) \). To get positive stuff, we need \( p > 3q \). To be primitive we need \( p, q \) to be relatively prime and \( p \) not divisible by 3. For example, if \( p = 7, q = 2 \) we get the triple \( (305, 1064, 35113) \).

There are other solutions to \( x^2 + 3y^2 = z^2 \), so this approach misses lots of stuff. Also, the triple most people started with, \( (1, 2, 3) \) can’t be found this way because \( x + 2y = 3 \), which isn’t a square (remember we were letting the triple by \( (x, 2y, z) \) to simplify the algebra, so with \( (1, 2, 3) \) \( y = 1 \)). That is, there are cases where \( x + 2y \) and \( x^2 - 2xy + 4y^2 \) have 3 as a common factor. **I’ll award extra credit** if someone can find (with proof) a simple infinite family of primitive solutions in this case where 3 is a common factor.

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**Solution:** I won’t do it, but this is not too difficult for \( x^2 + y^2 = z^3 \), or in fact, for \( x^2 + y^2 = z^n \) for any \( n \), since the Gaussian integers have unique factorization.