Periodic Behavior in a Class of Second Order Recurrence Relations Over the Integers

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Abstract

A recurrence relation is an equation that defines a sequence as a function of the preceding terms. The order of the recurrence is defined to be the number of previous terms needed to determine the next term in the sequence. A recurrence is linear if these terms are all to the first power and separate. When the resulting sequence repeats after k terms, we call the solution periodic with period k. Much is already known about the periodic behavior of linear recurrence relations. In this project we consider a nonlinear variation on a class of second order recurrence relations. The system that we looked at is

$$a_n = \begin{cases} x(Pa_{n-1} - Qa_{n-2}), & x(Pa_{n-1} - Qa_{n-2}) \in \mathbb{Z} \\ Pa_{n-1} - Qa_{n-2}, & \text{otherwise,} \end{cases}$$

where x is a rational number, P and Q are integers, and $\{a_n\}$ is the resulting sequence. We are interested in finding when periodic solutions occur. In other words, we want x values and initial conditions for specific values of P and Q which lead to a periodic solution. Using a common linear algebra method for solving recurrence relations, we developed a search method. In our search, P and Q were restricted such that $1 \leq P \leq 20$ and $-20 \leq Q \leq 20$ and $Q \neq 0$.

Contents

1 Introduction

A recurrence relation is an equation that defines a sequence as a function of the preceding terms. The order of the recurrence is defined to be the number of previous terms needed to determine the next term in the sequence. A recurrence is linear if these terms are all to the first power and separate. Therefore, a k^{th} order linear recurrence relation is an equation of the form: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$. The relation is homogeneous if $f(n) = 0$. Thus a first order linear homogeneous recurrence relation with constant coefficients has the form $a_n = c a_{n-1}$ where c is a constant. Here, the type of linear recurrence we are most concerned with is a second order of the form

$$a_n = P a_{n-1} - Q a_{n-2} \tag{1.1}$$

where initial conditions a_0 and a_1 are needed to determine a sequence and P and Q are constant coefficients. We use this form to be consistent with [5, p.44], [6, p.107] and [9].

The characteristic polynomial associated with this relation is $x^2 - Px + Q$. The roots of this polynomial are known as the characteristic roots and allow us to find the general solution to the system. The general solution has one of two possible forms depending on whether there are two distinct roots or one repeated root. If there are two distinct roots, the general solution will be of the form $a_n = Ac_1^n + Bc_2^n$ where c_1 and c_2 are the characteristic roots and A and B are constants. For example, when $a_n = 5a_{n-1} - 4a_{n-2}$ the characteristic polynomial is $x^2 - 5x + 4$. In this case, the characteristic roots are $x = 1$ and $x = 4$. Thus the general solution is $a_n = A4^n + B$. If the initial conditions are $a_0 = 1$ and $a_1 = 10$, then the solution is $a_n = 3 \cdot 4^n - 2$. If there is a single repeated root c, the general solution will have the form $a_n = (A + Bn)c^n$ again with A and B as constants. For instance, when $a_n = 6a_{n-1} - 9a_{n-2}$, the characteristic polynomial is $x^2 - 6x + 9$ which has a repeated root, $x = 3$. Therefore, the general solution is $a_n = (A + Bn)3^n$. If the initials conditions are $a_0 = 1$ and $a_1 = 9$, then the solution is $a_n = (1 + 2n)3^n$. More information on linear recurrence relations can be found in [2, Ch.10], [7, Ch.6] and [8, Ch.3.3].

A common Linear Algebra approach to solving recurrence relations is to convert (1.1) to a first order system, namely,

$$\boldsymbol{v}_n = \boldsymbol{A}\boldsymbol{v}_{n-1} \tag{1.2}$$

where $\boldsymbol{v}_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$. This method can be seen in numerous Linear Algebra textbooks, including [1, p.243] and [10, p.307]. We want $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \boldsymbol{v}_n = \boldsymbol{A}\boldsymbol{v}_{n-1} = \boldsymbol{A}\begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$. Thus, if we consider $\boldsymbol{A} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then we need $a_{n+1} = wa_n + xa_{n-1}$ and $a_n = ya_n + za_{n-1}$. In this case, these conditions are satisfied when $w = P$, $x = -Q$, $y = 1$, $z = 0$, so $\boldsymbol{A} = \begin{bmatrix} P & -Q \\ 1 & 0 \end{bmatrix}$. Notice the characteristic polynomials of (1.1) and $\boldsymbol{A}$ are the same. Thus the characteristic roots of the recurrence are the eigenvalues of $\boldsymbol{A}$. An advantage of this translation is that, by iteration, $\boldsymbol{v}_n = \boldsymbol{A}^n\boldsymbol{v}_0$.

The Lucas sequences U_n and V_n are specific sequences that satisfy (1.1) [5, p.41][6, p.107]. More specifically, for fixed P and Q, $U_n = PU_{n-1} - QU_{n-2}$ with $U_0 = 0$ and $U_1 = 1$. As before, if the characteristic roots, c_1 and c_2, are distinct, then $U_n = Ac_1^n + Bc_2^n$. Since U_n is defined to always start with the same initial conditions, we can use $U_0 = 0$ and $U_1 = 1$ to find that $U_n = \frac{c_1^n - c_2^n}{c_1 - c_2}$ is the general solution. In the previous example where $a_n = 5a_{n-1} - 4a_{n-2}$, the characteristic roots are $x = 4$ and $x = 1$. Thus $U_n = \frac{1}{3}(4^n - 1)$ when $P = 5$ and $Q = 4$. Perhaps the most famous example of a Lucas sequence is the Fibonacci sequence which satisfies the above equation with $P = 1$ and $Q = -1$.

The second Lucas sequence, V_n, is defined similarly by $V_n = PV_{n-1} - QV_{n-2}$. However the initial conditions are $V_0 = 2$ and $V_1 = P$. When $P = 1$ and $Q = -1$ the recurrence V_n yields the sequence 2, 1, 3, 4, 7, 11, … known as the Lucas numbers. Again, we can use the initial conditions and the form of the general solution to find V_n. When c_1 and c_2 are distinct characteristic roots, $V_n = c_1^n + c_2^n$. So, for example, when $P = 5$ and $Q = 4$, $V_n = 4^n + 1$.

A common problem with recurrence relations deals with the search for periodic solutions. When considering second order linear recurrences, frequently the only periodic solution is the zero sequence. A brief exploration of the periodic solutions of linear recurrences will be given later in this paper. Since the existence of periodic solutions to these recurrences is generally lacking or already known, a far more interesting problem deals with variations on the second order linear recurrence in (1.1). Consider the nonlinear recurrence

$$a_n = \begin{cases} x(a_{n-1} + a_{n-2}), & x(a_{n-1} + a_{n-2}) \in \mathbb{Z} \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases} \tag{1.3}$$

where x is a rational number. When looking for periodic solutions to (1.3), one must not only look at the initial conditions, but also the possible values of x. For example, consider

$$a_n = \begin{cases} \frac{1}{5}(a_{n-1} + a_{n-2}), & \frac{1}{5}(a_{n-1} + a_{n-2}) \in \mathbb{Z} \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases} \tag{1.4}$$

which is (1.3) when $x = \frac{1}{5}$. When starting with $a_0 = 3$ and $a_1 = 1$ as initial conditions, (1.4) has the periodic solution 3, 1, 4, 1, 1, 2, 3, 1, … The problem here is finding initial conditions and x values that allow for periodic solutions to appear. Previously, Niedzielski [3] investigated the existence of periodic solutions of (1.3).

Niedzielski [3] worked only with nonlinear adaptations of the Fibonacci numbers. However, periodic behavior is not limited to (1.3). For example, the relation

$$a_n = \begin{cases} \frac{1}{64}(2a_{n-1} - 6a_{n-2}), & \frac{1}{64}(2a_{n-1} - 6a_{n-2}) \in \mathbb{Z} \\ (2a_{n-1} - 6a_{n-2}), & \text{otherwise} \end{cases} \tag{1.5}$$

has a periodic solution with initial conditions $a_0 = 4$ and $a_1 = -13$, specifically 4, -13, -50, -22, 4, 140, 4, -13, … The generalization of this relation that was studied in this project is a nonlinear variation of the Lucas sequences, namely the system

$$a_n = \begin{cases} x(Pa_{n-1} - Qa_{n-2}), & x(Pa_{n-1} - Qa_{n-2}) \in \mathbb{Z} \\ Pa_{n-1} - Qa_{n-2}, & \text{otherwise,} \end{cases} \tag{1.6}$$

where x is a rational number and P and Q are integers. The question was when periodic behavior would occur. More precisely, given fixed values of P and Q, which values of x allow for periodic solutions?

This paper will adapt the methods in [3] used for investigating recurrence (1.3) to the more general setting in (1.6). Due to the increase in variables from (1.3) to (1.6), Niedzielski's method became difficult. Chapter 2 will include the essential theorems used to create the search method and reduce the number of cases to be examined. The approach taken will be outlined in Chapter 3.

The information in Chapter 2 and Chapter 3 made it possible to search for which pairs of P and Q had x values which allowed for periodic solutions. More specifically, the search included P and Q such that $1 \leq P \leq 20$ and $-20 \leq Q \leq 20$ and $Q \neq 0$. Chapter 4 contains the results from this search. This includes the values of P, Q and x where periodic solutions were found. Conjectures made about patterns in the data will be identified in Chapter 5. Suggestions for further work and open questions will be contained within Chapter 6.

2 Theoretical Considerations

A general problem with recurrence relations deals with the search for initial conditions that lead to periodic solutions. Below is a discussion of the periodic solutions of first and second order linear homogenous recurrence relations. Some of the results of this project follow similar patterns to those outlined below.

2.1 First Order Linear Recurrences

Recall the form of the first order linear recurrence, $a_n = ca_{n-1}$. Given initial condition a_0, the general solution is $a_n = a_0c^n$. The most trivial periodic solution to $a_n = ca_{n-1}$ would be $a_n = 0$. With the general solution in mind, we can easily see that trivial solution can be achieved when $a_0 = 0$ or $c = 0$. Slightly less trivial solutions would be nonzero constant solutions. If we want a_n to be some constant a for all n, we need $a = ca$. Since we are looking for nonzero solutions, we need $c = 1$. As a matter of fact, when $c = 1$, $a_n = a_{n-1}$ so every solution will be periodic with period one.

Consider possible solutions with longer period length. A solution of period two would be of the form $a, b, a, b, a, \ldots$ Thus, in order to achieve a solution of period two we need $a = cb$ and $b = ca$. By substitution, we get the condition $a = c^2a$. Again, we are looking for nonzero solutions, therefore we need $c^2 = 1$. As mentioned above, $c = 1$ leads to solutions with period one so we look at the case where $c = -1$. In this case, $a_n = -a_{n-1}$, thus all nonzero solutions will be periodic with period two.

Using the concept above, notice that solutions with longer period length must satisfy $a = c^ka$ where k is the length of the period. Therefore, when looking for periodic solutions, we are looking for values of c such that $c^k = 1$, values referred to as k^{th} roots of unity. Moreover, to ensure the period length is k, we need k to be the smallest positive integer for which $c^k = 1$. In this case, c is known as a primitive k^{th} root of unity. There is always a complex value of c for which $c^k = 1$. This value is $c =$

$e^{2\pi i/k}$ which by Euler's formula can be written as $c = e^{2\pi i/k} = \cos(2\pi/k) + i\sin(2\pi/k)$. In fact, all solutions are given by $c = e^{2\pi ij/k}$ where j is an integer. The roots will be primitive when j is relatively prime to k. If we are looking for real values of c we need $\sin(2\pi/k) = 0$. Thus $k = 1$ and $k = 2$ are the only possible solutions. Therefore, when considering first order linear homogeneous recurrence relations with constant coefficients, the only real periodic solutions that exist are of period length one or two. Moreover, when a sequence of period two exists, all nontrivial sequences to that recurrence will have period two.

2.2 Second Order Linear Recurrences

Although this project deals with a nonlinear variation of $a_n = Pa_{n-1} - Qa_{n-2}$, we will look briefly at the periodic solutions to this linear recurrence. Again, similar to the first order relation, when looking at periodic solutions of period length one there are the trivial solution $a_n = 0$ and the constant solution $a_n = a$. In the constant case we have $a = Pa - Qa$. Since we are looking for nonzero solutions we can assume $a \neq 0$ and divide by a to get $1 = P - Q$, or $P = Q + 1$. Therefore, the constant solution will be present when $P = Q + 1$ and $a_0 = a_1$. For example, when $P = 5$ and $Q = 4$, the initial conditions $a_0 = a_1 = 5$ result in the periodic solution $5, 5, 5, 5, \ldots$

Yet again we will explore larger period lengths. For a period of length two, we will have $a = Pb - Qa$ and $b = Pa - Qb$. If we add these together we obtain $a + b = P(a + b) - Q(a + b)$. There are two cases: either $a + b \neq 0$ or $a + b = 0$. If $a + b \neq 0$ then we can divide by $a + b$ and we have the condition $P = Q + 1$ again. Substituting this condition in to $a = Pb - Qa$ yields $(Q + 1)a = (Q + 1)b$. Since $a \neq b$, this implies $Q + 1 = 0$. Therefore, $Q = -1$ and $P = 0$. This gives us the recurrence $a_n = a_{n-2}$, which is always periodic. In order to assure a period length of two we must specify $a_0 \neq a_1$, otherwise the period would be one. If $a + b = 0$ then $b = -a$. Again we substitute this in to $a = Pb - Qa$ to receive $a = -Pa - Qa$ so $P = -Q - 1$. When both of these conditions hold, all solutions will be periodic with period two. For instance, if we choose $Q = -4$ then $P = 3$ so $a_n = 3a_{n-1} + 4a_{n-2}$. Starting with the initial conditions $a_0 = 1$ and $a_1 = -1$, we have the periodic solution $1, -1, 1, -1, 1, \ldots$ In this case, however, we will not get periodic behavior when $-a_0 \neq a_1$. For example, with the same recurrence, $a_n = 3a_{n-1} + 4a_{n-2}$, if the initial conditions are instead $a_0 = 1$ and $a_1 = 4$ we have the non-periodic solution $1, 4, 16, 64, \ldots$

For larger periods, at least one of the roots of the characteristic polynomial must be a k^{th} root of unity. Similar to the first order case, we know $c_1 = e^{2\pi ij/k}$ will work. Since complex solutions must come in conjugate pairs, in order for P and Q to be real, our second solution must be $c_2 = e^{-2\pi ij/k}$. With these solutions, P must be $c_1 + c_2$ so $P = 2\cos(2\pi j/k)$ and $Q = c_1 \cdot c_2 = 1$. This gives the relation $a_n = 2\cos(2\pi j/k)\, a_{n-1} - a_{n-2}$ for some j relatively prime to k. It turns out that every solution will be periodic with a period length which divides k. To see why this is true, let $\omega = e^{2\pi ij/k}$ then ω and ω^{-1} are the roots of the characteristic polynomial. Thus the general solution is $a_n = A\omega^n + B\omega^{-n}$ for some constants A and B. Since $\omega^k = 1$ we have $a_{n+k} = A\omega^{n+k} + B\omega^{-n-k} = A\omega^n\omega^k + B\omega^{-n}\omega^{-k} = A\omega^n + B\omega^{-n} = a_n$. In this project, we focus on integer sequences. Since $a_n = 2\cos(2\pi j/k)\, a_{n-1} - a_{n-2}$, we know $a_n + a_{n-2} = 2\cos(2\pi j/k)\, a_{n-1}$. If we are only interested in integer sequences, this implies $2\cos(2\pi j/k)$ must be a rational number.

Niven's Theorem: If θ is rational in degrees, say $\theta = 2\pi r$ for some rational number r, then the only rational values of the trigonometric functions of θ are as follows: $\sin\theta, \cos\theta = 0, \pm\frac{1}{2}, \pm 1$; $\sec\theta, \csc\theta = \pm 1, \pm 2$; $\tan\theta, \cot\theta = 0, \pm 1$. [4, p.41]

Consequently, the only angles which will make $2\cos(2\pi j/k)$ a rational number are $\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, 2\pi$. Therefore, the only values for k which allow for periodic solutions of period k are $k = 1, 2, 3, 4, 6$. For example, when $k = 4$, $2\cos(2\pi j/k) = 2\cos(2\pi j/4) = 2\cos(\pi j/2) = 0$ so $a_n = -a_{n-2}$. In this case, all nontrivial solutions will have period four. For instance, when $a_0 = 1$ and $a_1 = 4$ the solution will be periodic with period four, specifically $1, 4, -1, -4, 1, 4, \ldots$ Similarly, when $k = 6$,

$$\begin{aligned} a_n &= 2\cos(2\pi j/6)\, a_{n-1} - a_{n-2} \\ &= 2\cos(\pi j/3)\, a_{n-1} - a_{n-2} \\ &= 2\left(\frac{1}{2}\right) a_{n-1} - a_{n-2} \\ &= a_{n-1} - a_{n-2}. \end{aligned}$$

Here, all nontrivial solutions will have period six. One such example occurs when $a_0 = 3$ and $a_1 = 4$ where the solution is $3, 4, 1, -3, -4, -1, 3, 4, \ldots$

As mentioned previously, for this project, we restricted our search to only positive values of P. We were able to do this because if $a_n = f(n)$ is a solution to $a_n = -Pa_{n-1} - Qa_{n-2}$ then $b_n = (-1)^n f(n)$ is a solution to $b_n = Pb_{n-1} - Qb_{n-2}$. To see this, let $a_n = f(n)$ be a solution to $a_n =$

$-Pa_{n-1} - Qa_{n-2}$. Then $f(n) = -Pf(n-1) - Qf(n-2)$. Let $b_n = (-1)^n f(n)$ so $f(n) = (-1)^n b_n$. Therefore, by substitution, $(-1)^n b_n = P(-1)^{n-1} b_{n-1} - Q(-1)^{n-2} b_{n-2}$. If we divide by $(-1)^n$, we see that $b_n = Pb_{n-1} - Qb_{n-2}$. Thus $b_n = (-1)^n f(n)$ is a solution to $b_n = Pb_{n-1} - Qb_{n-2}$ when $a_n = f(n)$ is a solution to $a_n = -Pa_{n-1} - Qa_{n-2}$. Therefore, we know for any solution we find for positive P, there is also a periodic solution in the negative P case where the period is half that of the positive case. For example, the recurrence $a_n = -a_{n-1} - a_{n-2}$ when $a_0 = 5$ and $a_1 = 3$ has periodic solution $5, 3, -8, 5, 3, \ldots$ This is the case of (1.1) where $P = -1$ and $Q = 1$. Consider what happens when $P = 1$ and $Q = 1$. In this instance, $a_n = a_{n-1} - a_{n-2}$. When the initial conditions are $a_0 = 5$ and $a_1 = -3$, this recurrence has the periodic solution $5, -3, -8, -5, 3, 8, 5, -3, \ldots$ As one can see, the period length of the positive case is merely double that of the case where P is negative.

2.3 A Nonlinear Variation on Second Order Recurrences

In the search for periodic solutions to (1.3), Niedzielski [3] used the method outlined below to find values of x which would allow for periodic solutions. The approach used involved a shift to a first order system similar to (1.2). Specifically,

$$\boldsymbol{v}_n = \begin{cases} \boldsymbol{B}\boldsymbol{v}_{n-1}, & x(a_n + a_{n-1}) \in \mathbb{Z} \\ \boldsymbol{A}\boldsymbol{v}_{n-1}, & \text{otherwise} \end{cases} \tag{2.1}$$

where $\boldsymbol{v}_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$ with a_n and a_{n+1} as sequence terms, $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $\boldsymbol{B} = \begin{bmatrix} x & x \\ 1 & 0 \end{bmatrix}$. To see the benefit of this transition, consider the example of (1.4) from the introduction. The periodic solution was 3, 1, 4, 1, 1, 2, 3, 1, … This corresponds to $\boldsymbol{v}_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \boldsymbol{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \boldsymbol{v}_5 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \boldsymbol{v}_6 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \cdots$ when we use (2.1). Since $\boldsymbol{v}_6 = \boldsymbol{B}\boldsymbol{v}_5$, $\boldsymbol{v}_5 = \boldsymbol{A}\boldsymbol{v}_4$, $\boldsymbol{v}_4 = \boldsymbol{A}\boldsymbol{v}_3$, and so on, we see that $\boldsymbol{v}_0 = \boldsymbol{B}\boldsymbol{A}^2\boldsymbol{B}^2\boldsymbol{A}\boldsymbol{v}_0$. That is, $\boldsymbol{v}_0$ is an eigenvector of $\boldsymbol{B}\boldsymbol{A}^2\boldsymbol{B}^2\boldsymbol{A}$ with eigenvalue 1.

More generally, a periodic solution to (2.1) of length m means $\boldsymbol{v}_{n+m} = \boldsymbol{v}_n$ for all n. Therefore, if the period length is m, $\boldsymbol{v}_m = \boldsymbol{v}_0$. However, the solution to the recurrence is $\boldsymbol{v}_m = \boldsymbol{W}\boldsymbol{v}_0$ where $\boldsymbol{W}$ is a product of m $\mathbf{A}$*'s* and $\mathbf{B}$*'s.* This leads to the following:

Theorem 2.1: For every periodic solution to (1.3), there is a corresponding matrix $\boldsymbol{W}$, which is a product of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s, that has an eigenvector $\boldsymbol{v}_0$, with eigenvalue 1. The entries of $\boldsymbol{v}_0$ give the initial conditions for the periodic solution.

Consider how this would change for our more generalized problem. In (1.5) from the introduction, we no longer have one as coefficients. Thus multiplying by $\boldsymbol{A}$ and $\boldsymbol{B}$ as defined above will not give us the desired result. Instead, (1.5) can be considered as

$$\boldsymbol{v}_n = \begin{cases} \boldsymbol{B}\boldsymbol{v}_{n-1}, & \dfrac{1}{64}(2a_n - 6a_{n-1}) \in \mathbb{Z} \\ \boldsymbol{A}\boldsymbol{v}_{n-1}, & \text{otherwise} \end{cases} \tag{2.2}$$

where $\boldsymbol{v}_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$ still has a_n and a_{n+1} as sequence terms. However, now $\boldsymbol{A} = \begin{bmatrix} 2 & -6 \\ 1 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} \frac{2}{64} & \frac{-6}{64} \\ 1 & 0 \end{bmatrix}$ where $\boldsymbol{A}$ and $\boldsymbol{B}$ are determined in the same manner as (1.2). With these specifications, the periodic solution 4, -13, -50, -22, 4, 140, 4, -13, … can now be written as $\boldsymbol{v}_0 = \begin{bmatrix} -13 \\ 4 \end{bmatrix}, \boldsymbol{v}_1 = \begin{bmatrix} -50 \\ -13 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -22 \\ -50 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 4 \\ -22 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} 140 \\ 4 \end{bmatrix}, \boldsymbol{v}_5 = \begin{bmatrix} 4 \\ 140 \end{bmatrix}, \boldsymbol{v}_6 = \begin{bmatrix} -13 \\ 4 \end{bmatrix}, \cdots$ In this case, we see that $\boldsymbol{v}_0 = \boldsymbol{B}^2\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^2\boldsymbol{v}_0$. Observe that the coefficients 2 and 6 correspond to P and Q in our generalization. This leads to the following:

Theorem 2.2: For every periodic solution to (1.6), there is a corresponding matrix $\boldsymbol{W}$, which is a product of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s, that has an eigenvector $\boldsymbol{v}_0$, with 1 as an eigenvalue. Where the entries of $\boldsymbol{v}_0$ give the initial conditions for the periodic solution of the system

$$\boldsymbol{v}_n = \begin{cases} \boldsymbol{B}\boldsymbol{v}_{n-1}, & x(Pa_n - Qa_{n-1}) \in \mathbb{Z} \\ \boldsymbol{A}\boldsymbol{v}_{n-1}, & \text{otherwise.} \end{cases} \tag{2.3}$$

Where $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined as $\boldsymbol{A} = \begin{bmatrix} P & -Q \\ 1 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} Px & -Qx \\ 1 & 0 \end{bmatrix}$. Recall that x is a rational number and P and Q are the integer coefficients from (1.6).

With this in mind, we consider how to find the products of $\boldsymbol{A}$ and $\boldsymbol{B}$ which will have 1 as an eigenvalue. It is known that a matrix $\boldsymbol{W}$ has eigenvalue 1 if and only if $\det(\boldsymbol{W} - \boldsymbol{I}) = 0$. For a two by two matrix, the characteristic polynomial simplifies to the particularly nice form $\det(\boldsymbol{W} - \boldsymbol{I}) = 1 -$

$\operatorname{tr} \boldsymbol{W} + \det \boldsymbol{W}$. Now let $\boldsymbol{W}$ be some product of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ from above. We can define the function

$$f_W(x) = -\det(\boldsymbol{W} - \boldsymbol{I}) = \operatorname{tr} \boldsymbol{W} - \det \boldsymbol{W} - 1. \tag{2.4}$$

Notice that instead of using $\det(\boldsymbol{W} - \boldsymbol{I})$, $f_W(x)$ is the negative of the characteristic polynomial. This was done to make the leading coefficients in (3.1), (3.2), (3.3), and (3.4) look nicer.

Thus $\boldsymbol{W}$ will have 1 as an eigenvalue for exactly those x values that satisfy $f_W(x) = 0$. Therefore, when given a specific product of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s, where P and Q are fixed, we can find all x values with 1 as an eigenvalue by solving $f_W(x) = 0$. For example, consider (2.2) from above. Recall $P = 2$ and $Q = 6$, thus $\boldsymbol{A} = \begin{bmatrix} 2 & -6 \\ 1 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 2x & -6x \\ 1 & 0 \end{bmatrix}$. Therefore,

$$\boldsymbol{W} = \boldsymbol{B}^2\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^2 = \begin{bmatrix} 64x^3 + 240x^2 - 72x & -48x^3 + 144x^2 - 432x \\ 32x^2 + 24x & -24x^2 + 144x \end{bmatrix}.$$

With some simple calculations, we see that $\operatorname{tr} \boldsymbol{W} = 64x^3 + 240x^2 - 72x$ and $\det \boldsymbol{W} = 46656x^3$, thus

$$\begin{aligned} f_W(x) &= 64x^3 + 240x^2 - 72x - 46656x^3 - 1 \\ &= -46592x^3 + 216x^2 + 72x - 1 \\ &= (64x - 1)(-728x^2 - 8x + 1). \end{aligned}$$

From here, it is easy to see that $x = \frac{1}{64}$ is a root. If we substitute this solution into our matrix, we get

$$\boldsymbol{W} = \begin{bmatrix} -\dfrac{4367}{4096} & -\dfrac{110019}{16384} \\ \dfrac{49}{128} & \dfrac{1149}{512} \end{bmatrix}.$$

We know this matrix has an eigenvalue of 1. If we solve for the corresponding eigenvector, we obtain $\boldsymbol{v}_0 = \begin{bmatrix} -13 \\ 4 \end{bmatrix}$, which gives the initial condition leading to the periodic solution addressed above. However, as is, there are a couple of factors that we must consider before we can adopt this as the search method when looking for periodic solutions to (1.6).

Unfortunately, Theorem 2.2 does not guarantee periodic behavior. In other words, it is possible to have a solution to $f_W(x) = 0$ that does not lead to a periodic solution. For example, consider (2.3) where $P = 4$ and $Q = 6$. In this case,

$$\boldsymbol{W} = \boldsymbol{B}\boldsymbol{A}^2 = \begin{bmatrix} 16x & -60x \\ 10 & -24 \end{bmatrix}.$$

Thus

$$\begin{aligned} f_W(x) &= 16x - 24 - 216x - 1 \\ &= -200x - 25. \end{aligned}$$

Therefore, the matrix $\boldsymbol{W}$ has an eigenvalue of 1 when $x = -\frac{1}{8}$. In this case, $\boldsymbol{W} = \begin{bmatrix} -2 & \frac{15}{2} \\ 10 & -24 \end{bmatrix}$ and the corresponding eigenvector is $\boldsymbol{v}_0 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. However, this leads to the sequence $2, 5, -1, -34, -130, -316, \ldots$ which does not have the expected periodic behavior. Typically, the problem in these cases is a division in the wrong place. In this case, a division occurred right away and then not when expected. If we were to not allow the division immediately after the initial conditions, the solution would be periodic. In other words, if $a_0 = 2$ and $a_1 = 5$, then $a_n = 4a_{n-1} - 6a_{n-2}$ gives us $a_2 = 8$. This time, we don't multiply by $x = -\frac{1}{8}$. Instead, we use our linear recurrence again to get $a_3 = 2$. For a_4, $a_n = 4a_{n-1} - 6a_{n-2}$ yields -40 which is divisible by -8. We allow the division and thus have the periodic solution $2, 5, 8, 2, 5, \ldots$ with period three as expected. There were several other scenarios in which this occurred. For instance, when $P = 4$ and $Q = -18$ the matrix $\boldsymbol{W} = \boldsymbol{B}\boldsymbol{A}^2\boldsymbol{B}$ has a similar outcome. Here,

$$\boldsymbol{W} = \begin{bmatrix} 832x^2 + 612x & 3744x^2 \\ 136x + 72 & 612x \end{bmatrix}$$

so

$$\begin{aligned} f_W(x) &= 832x^2 + 1224x - 104976x^2 - 1 \\ &= -104144x^2 + 1224x - 1 \\ &= (92x - 1)(-1132x + 1). \end{aligned}$$

The value $x = \frac{1}{92}$ is one of the solutions to $f_W(x) = 0$ that guarantees $\boldsymbol{W}$ has an eigenvalue of 1. The eigenvector with $x = \frac{1}{92}$ is $\boldsymbol{v}_0 = \begin{bmatrix} -1 \\ 13 \end{bmatrix}$. Using (2.3), we get the sequence

$13, -1, 230, 902, 7748, 47228, \ldots$ which is, once again, not periodic. Here, it turns out that while $\boldsymbol{v}_0 = \begin{bmatrix} -1 \\ 13 \end{bmatrix}$ is an eigenvector, it does not lead to periodic behavior. If we instead started with $2\boldsymbol{v}_0 = \begin{bmatrix} -2 \\ 26 \end{bmatrix}$, the resulting sequence would be $26, -2, 5, -16, 26, -2, \ldots$ which is periodic as expected. The second x value, $x = \frac{1}{1132}$, also did not lead to periodic behavior because of the same problem. There are also cases where one x value leads to periodic behavior, but the second x value does not. For instance, when $P = 5$ and $Q = -7$, the matrix $(\boldsymbol{BA}^3)^2$ has two x values that guarantee an eigenvalue of 1. These values are $x = \frac{223}{1202}$ and $x = -\frac{1}{16}$. The first, $x = \frac{223}{1202}$, has corresponding eigenvector $\begin{bmatrix} -223 \\ 195 \end{bmatrix}$ which leads to the periodic solution $195, -223, 250, -311, 195, -223, \ldots$ However, the eigenvector, $\begin{bmatrix} -15 \\ 13 \end{bmatrix}$, corresponding to the second solution, $x = -\frac{1}{16}$, does not lead to a periodic solution. The resulting sequence is $13, -15, -1, -110, -557, -3555, \ldots$ This was yet another case where the multiplications by x did not follow the expected pattern. Based on these cases, it is clear to see that each result must be tested to verify that the solution was indeed periodic.

Another difficulty is the number of possible cases. To begin with, even if we limit the length of products of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s to 24, we would still have over 16 million possible products. In addition to that, we are interested in multiple values of P and Q for each product of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s. As mentioned in the introduction, we limited the possible values of P and Q to the range $1 \leq P \leq 20$ and $-20 \leq Q \leq 20$. This still leaves nearly 800 possible pairs of P's and Q's for each of the 16 million possible products of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s. In other words, the number of possible cases that we would currently need to check is daunting at best. In an attempt to minimize the number of cases that must be considered, we look at the order in which the $\boldsymbol{A}$'s and $\boldsymbol{B}$'s appear in our products. Looking at equation (2.4), we note that the order of our product will only affect the equation in the same manner that the determinant and trace are affected by order. The determinant of $\boldsymbol{W}$ is not dependent on the order of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s since $\det \boldsymbol{AB} = \det \boldsymbol{A} \det \boldsymbol{B} = \det \boldsymbol{B} \det \boldsymbol{A} = \det \boldsymbol{BA}$. Therefore, the only term in (2.4) that depends on the order of the product of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s is $\operatorname{tr} \boldsymbol{W}$.

Next, it is known that cyclic permutations of matrix products have the same trace. Therefore, any cyclic permutations of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s will have the same periodic solution. For example, consider (2.2) from above. In this example, $P = 2$, $Q = 6$ and $x = \frac{1}{64}$. A periodic solution begins with $a_0 = 4$, $a_1 = -13$, corresponding to $v_0 = \begin{bmatrix} -13 \\ 4 \end{bmatrix}$, an eigenvector of $\boldsymbol{B}^2\boldsymbol{ABA}^2$ with eigenvalue 1. However, $\boldsymbol{v}_0 =$

$\boldsymbol{B}^2\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^2\boldsymbol{v}_0$ is not the only possibility; if we were to pick any two consecutive terms in the periodic solution, they could be used to form a periodic solution with the same numbers, but different initial conditions. However, our product of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s would differ. Suppose an initial condition of $\begin{bmatrix} -50 \\ -13 \end{bmatrix}$ is used instead of $\begin{bmatrix} -13 \\ 4 \end{bmatrix}$. Our periodic solution now corresponds to $\boldsymbol{v}_0 = \begin{bmatrix} -50 \\ -13 \end{bmatrix}, \boldsymbol{v}_1 = \begin{bmatrix} -22 \\ -50 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 4 \\ -22 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 140 \\ 4 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} 4 \\ 140 \end{bmatrix}, \boldsymbol{v}_5 = \begin{bmatrix} -13 \\ 4 \end{bmatrix}, \boldsymbol{v}_6 = \begin{bmatrix} -50 \\ -13 \end{bmatrix} \cdots$ Here we see that $\boldsymbol{v}_6 = \boldsymbol{A}\boldsymbol{v}_5$, $\boldsymbol{v}_5 = \boldsymbol{B}\boldsymbol{v}_4$, $\boldsymbol{v}_4 = \boldsymbol{B}\boldsymbol{v}_3$, … which leads us to $\boldsymbol{v}_0 = \boldsymbol{A}\boldsymbol{B}^{\boldsymbol{2}}\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}\boldsymbol{v}_0$. Recall that when $\boldsymbol{v} = \begin{bmatrix} -13 \\ 4 \end{bmatrix}$ the product of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s which resulted was $\boldsymbol{v} = \boldsymbol{B}^2\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^2\boldsymbol{v}$. Notice that these are merely cyclic permutations of each other. Therefore, cyclic permutations of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s will result in the same periodic solution only shifted depending on the initial conditions. Below is the proof for the two by two case that the trace does not change for cyclically permuted products.

Theorem 2.3: Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n$ be 2x2 matrices. Then

$$\operatorname{tr}(\boldsymbol{A}_1\, \boldsymbol{A}_2 \cdot \cdots \cdot \boldsymbol{A}_n) = \operatorname{tr}(\boldsymbol{A}_n\, \boldsymbol{A}_1 \cdot \cdots \cdot \boldsymbol{A}_{n-1}).$$

Proof: Let $\boldsymbol{A}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\boldsymbol{A}_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then

$$\boldsymbol{A}_1\boldsymbol{A}_2 = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix},$$

so

$$\operatorname{tr}(\boldsymbol{A}_1\boldsymbol{A}_2) = ae + bg + cf + dh.$$

Also,

$$\boldsymbol{A}_2\boldsymbol{A}_1 = \begin{bmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{bmatrix}$$

has

$$\operatorname{tr}(\boldsymbol{A}_2\boldsymbol{A}_1) = ea + fc + gb + hd.$$

Therefore $\operatorname{tr}(\boldsymbol{A}_1\boldsymbol{A}_2) = \operatorname{tr}(\boldsymbol{A}_2\boldsymbol{A}_1)$. In general, for matrices $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n$, let

$$\boldsymbol{B} = \boldsymbol{A}_1\, \boldsymbol{A}_2 \cdot \cdots \cdot \boldsymbol{A}_{n-1}$$

Then by the above calculations,

$$\operatorname{tr}(\boldsymbol{B}\boldsymbol{A}_n) = \operatorname{tr}(\boldsymbol{A}_n\boldsymbol{B})$$

or

$$\operatorname{tr}(\boldsymbol{A}_1\, \boldsymbol{A}_2 \cdot \cdots \cdot \boldsymbol{A}_n) = \operatorname{tr}(\boldsymbol{A}_n\, \boldsymbol{A}_1 \cdot \cdots \cdot \boldsymbol{A}_{n-1}).$$

Thus the trace is invariant under cyclic permutations. ■

Corollary 2.4*:* Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n$ be 2x2 matrices of the form $\begin{bmatrix} P & -Q \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} Px & -Qx \\ 1 & 0 \end{bmatrix}$. If

$\boldsymbol{W}_1 = \boldsymbol{A}_1\, \boldsymbol{A}_2 \cdot \cdots \cdot \boldsymbol{A}_n$ and $\boldsymbol{W}_2 = \boldsymbol{A}_n\, \boldsymbol{A}_1 \cdot \cdots \cdot \boldsymbol{A}_{n-1}$, then $f_{\boldsymbol{W}_1}(x) = f_{\boldsymbol{W}_2}(x)$.

Proof: By (2.4) and Theorem 2.3,

$$\begin{aligned} f_{\boldsymbol{W}_1}(x) &= \mathrm{tr}(\boldsymbol{A}_1\, \boldsymbol{A}_2 \cdot \cdots \cdot \boldsymbol{A}_n) - 1 - \det(\boldsymbol{A}_1\, \boldsymbol{A}_2 \cdot \cdots \cdot \boldsymbol{A}_n) \\ &= \mathrm{tr}(\boldsymbol{A}_n\, \boldsymbol{A}_1 \cdot \cdots \cdot \boldsymbol{A}_{n-1}) - 1 - \det(\boldsymbol{A}_n\, \boldsymbol{A}_1 \cdot \cdots \cdot \boldsymbol{A}_{n-1}) \\ &= f_{\boldsymbol{W}_2}(x). \quad \blacksquare \end{aligned}$$

Fortunately, this reduces the number of cases that must be looked at by a factor of about n when dealing with strings of length n.

As an example of Corollary 2.4, we saw two different matrices that led to periodic solutions for (2.2). Let us examine $f_W(x)$ in both of these cases. In the first example, $\boldsymbol{W}_1 = \boldsymbol{B}^2\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^2$ so

$$\boldsymbol{W}_1 = \begin{bmatrix} 64x^3 + 240x^2 - 72x & -48x^3 + 144x^2 - 432x \\ 32x^2 + 24x & -24x^2 + 144x \end{bmatrix}.$$

From here, we can easily determine the trace and determinant of $\boldsymbol{W}_1$. As a result,

$$\begin{aligned} \mathrm{tr}\, \boldsymbol{W}_1 &= 64x^3 + 240x^2 - 72x - 24x^2 + 144x \\ &= 64x^3 + 216x^2 + 72x \end{aligned}$$

and

$$\begin{aligned} \det \boldsymbol{W}_1 &= (64x^3 + 240x^2 - 72x)(-24x^2 + 144x) - (-48x^3 + 144x^2 - 432x)(32x^2 + 24x) \\ &= 46656x^3. \end{aligned}$$

In the second example, $\boldsymbol{W}_2 = \boldsymbol{A}\boldsymbol{B}^{\boldsymbol{2}}\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}.$ Thus

$$\boldsymbol{W}_2 = \begin{bmatrix} 16x^3 - 72x^2 + 288x & 96x^3 + 432x^2 - 864x \\ 8x^3 - 24x^2 + 72x & 48x^3 + 288x^2 - 216x \end{bmatrix}.$$

Again, we find the trace and determinant as follows:

$$\begin{aligned} \mathrm{tr}\, \boldsymbol{W}_2 &= 16x^3 - 72x^2 + 288x + 48x^3 + 288x^2 - 216x \\ &= 64x^3 + 216x^2 + 72x \end{aligned}$$

and

$$\det \boldsymbol{W}_2 = (16x^3 - 72x^2 + 288x)(48x^3 + 288x^2 - 216x)$$
$$- (96x^3 + 432x^2 - 864x)(8x^3 - 24x^2 + 72x)$$
$$= 46656x^3.$$

Note that $\boldsymbol{W}_1 \neq \boldsymbol{W}_2$. However, as expected, $f_{\boldsymbol{W}_1}(x) = f_{\boldsymbol{W}_2}(x)$ since $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ have the same trace and determinant.

Now we focus on searching for periodic solutions. Instead of attempting an exhaustive search by looking at the total number of matrices multiplied together, consider products based on how many times the matrix $\boldsymbol{B}$ appears in the complete product. Note, without loss of generality, we can represent any product of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s as the product of matrices of the form of $\boldsymbol{B}\mathbf{A}^{\mathrm{n}}$. For example, $\boldsymbol{AAABABBAABA}$ can be replaced by $\boldsymbol{BA}^1\boldsymbol{BA}^0\boldsymbol{BA}^2\boldsymbol{BA}^4$ by cyclically reordering the $\boldsymbol{A}$'s and $\boldsymbol{B}$'s. In other words, $\boldsymbol{W} = \boldsymbol{BA}^{n_1}\boldsymbol{BA}^{n_2} \ldots \boldsymbol{BA}^{n_k}$. It is convenient to write $\boldsymbol{W}$ in this form because of the facts below.

Lemma 2.5*:* If $\boldsymbol{A} = \begin{bmatrix} P & -Q \\ 1 & 0 \end{bmatrix}$, then $\boldsymbol{A}^n = \begin{bmatrix} U_{n+1} & -QU_n \\ U_n & -QU_{n-1} \end{bmatrix}$ where $\{U_n\}$ is the Lucas sequence described in Chapter 1*.*

Proof: If $\boldsymbol{A} = \begin{bmatrix} P & -Q \\ 1 & 0 \end{bmatrix}$, then $\boldsymbol{A}^2 = \begin{bmatrix} P^2 - Q & -PQ \\ P & -Q \end{bmatrix}$ and $\boldsymbol{A}^3 = \begin{bmatrix} P^3 - 2PQ & -P^2Q + Q^2 \\ P^2 - Q & -PQ \end{bmatrix}$. Assume that

$$\boldsymbol{A}^n = \begin{bmatrix} U_{n+1} & -QU_n \\ U_n & -QU_{n-1} \end{bmatrix}$$

for some integer n. Then

$$\boldsymbol{A}^{n+1} = \begin{bmatrix} P & -Q \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} U_{n+1} & -QU_n \\ U_n & -QU_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} PU_{n+1} - QU_n & -PQU_n + Q^2U_{n-1} \\ U_{n+1} & -QU_n \end{bmatrix}.$$

But

$$U_{n+2} = PU_{n+1} - QU_n \text{ and } -QU_{n+1} = -Q(PU_n - QU_{n-1})$$

so

$$\boldsymbol{A}^{n+1} = \begin{bmatrix} U_{n+2} & -QU_{n+1} \\ U_{n+1} & -QU_n \end{bmatrix}. \text{ Thus } \boldsymbol{A}^n = \begin{bmatrix} U_{n+1} & -QU_n \\ U_n & -QU_{n-1} \end{bmatrix}. \quad \blacksquare$$

Corollary 2.6: If $\mathbf{A} = \begin{bmatrix} \mathrm{P} & -\mathrm{Q} \\ 1 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} Px & -Qx \\ 1 & 0 \end{bmatrix}$, then $\boldsymbol{B}\mathbf{A}^{\mathrm{n}} = \begin{bmatrix} x\mathrm{U}_{\mathrm{n}+2} & -\mathrm{Q}x\mathrm{U}_{\mathrm{n}+1} \\ \mathrm{U}_{\mathrm{n}+1} & -\mathrm{QU}_n \end{bmatrix}$.

Proof: By Lemma 2.5,

$$\boldsymbol{B}\mathrm{A}^{\mathrm{n}} = \begin{bmatrix} Px & -Qx \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} \mathrm{U}_{\mathrm{n+1}} & -\mathrm{QU}_{\mathrm{n}} \\ \mathrm{U}_{\mathrm{n}} & -\mathrm{QU}_{\mathrm{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} Px\mathrm{U}_{\mathrm{n+1}} - Qx\mathrm{U}_{\mathrm{n}} & -PQx\mathrm{U}_{\mathrm{n}} + Q^2x\mathrm{U}_{\mathrm{n-1}} \\ \mathrm{U}_{\mathrm{n+1}} & -Q\mathrm{U}_{\mathrm{n}} \end{bmatrix}$$

$$= \begin{bmatrix} x\mathrm{U}_{\mathrm{n+2}} & -\mathrm{Q}x\mathrm{U}_{\mathrm{n+1}} \\ \mathrm{U}_{\mathrm{n+1}} & -\mathrm{QU}_{n} \end{bmatrix}.$$

Lemma 2.7: $U_n{}^2 - U_{n+1}U_{n-1} = Q^{n-1}$

Proof: Given $\boldsymbol{A}^n = \begin{bmatrix} U_{n+1} & -QU_n \\ U_n & -QU_{n-1} \end{bmatrix}$, $-QU_{n+1}U_{n-1} + Q\mathrm{U}_{\mathrm{n}}{}^2 = \det(\boldsymbol{A}^n) = (\det \boldsymbol{A})^n = Q^n$.

thus $\mathrm{U}_{\mathrm{n}}{}^2 - \mathrm{U}_{\mathrm{n+1}}\mathrm{U}_{\mathrm{n-1}} = \mathrm{Q}^{\mathrm{n-1}}$. ■

3 The Polynomial $f_W(x)$ When W Has a Small Number of B's

From Corollary 2.6, we have a formula for $\boldsymbol{BA}^n$. By Corollary 2.4 this is the only product of length $n+1$ with a single $\boldsymbol{B}$ that we must consider. In other words, if we are concerned with the product $\boldsymbol{A}^2\boldsymbol{B}\boldsymbol{A}^3$, we know from Chapter 2 that $f_{\boldsymbol{A}^2\boldsymbol{B}\boldsymbol{A}^3}(x) = f_{\boldsymbol{B}\boldsymbol{A}^3\boldsymbol{A}^2}(x) = f_{\boldsymbol{B}\boldsymbol{A}^5}(x)$. More generally, as mentioned previously, we can write any $\boldsymbol{W}$ in the form $\boldsymbol{B}\boldsymbol{A}^{n_1}\boldsymbol{B}\boldsymbol{A}^{n_2} \dots \boldsymbol{B}\boldsymbol{A}^{n_k}$.

As we focus on $f_W(x)$, it is important to note some commonly occurring solutions. There are several cases where $x = 1$, $x = -1$ or $x = 0$ might arise. It is easy to classify all cases of P and Q in which $x = 1$ will occur. This is because, when $x = 1$, $\boldsymbol{A} = \boldsymbol{B}$, so $\boldsymbol{W} = \boldsymbol{B}\boldsymbol{A}^{n_1}\boldsymbol{B}\boldsymbol{A}^{n_2} \dots \boldsymbol{B}\boldsymbol{A}^{n_k} = \boldsymbol{A}^m$ where $m = k + n_1 + n_2 + \cdots + n_k$. We know if c_1 and c_2 are eigenvalues of $\boldsymbol{A}$ then c_1^m and c_2^m are eigenvalues of $\boldsymbol{A}^m$ [1, p.250]. Since $\boldsymbol{A}^m$ must have an eigenvalue of 1, at least one of c_1 and c_2 must be a root of unity. In other words, suppose, without loss of generality, $c_1^m = 1$. This brings us back to the cases discussed in Section 2.2. There are three possible scenarios. Either $c_1 = 1$, $c_1 = -1$ or c_1 is complex. When $c_1 = 1$, $1 - P + Q = 0$ thus $P = Q + 1$. When $c_1 = -1$, $P = -Q - 1$. Finally, if c_1 is complex, c_2 must be it's conjugate pair. As a result, $P = c_1 + c_2 = 2\cos(2\pi j/k)$ and $Q = c_1 \cdot c_2 = 1$. These lead to the cases $P = 0$, $P = 1$ and $P = -1$, all discussed in Section 2.2. To summarize, if $\boldsymbol{W} = \boldsymbol{A}^m$, then $f_W(x)$ has $x = 1$ as a zero if and only if one of the following holds:

c_1	P	Q	m
1	arbitrary	$P-1$	no restrictions
-1	arbitrary	$-P-1$	even
$e^{2\pi i/3}$	-1	1	divisible by 3
i	0	1	divisible by 4
$e^{\pi i/3}$	1	1	divisible by 6

Unlike $x = 1$, it is difficult to characterize the cases where $x = -1$ or $x = 0$. More information on the consequence of these values will be addressed in Chapter 4.

Since we can write $\boldsymbol{W} = \boldsymbol{B}\boldsymbol{A}^{n_1}\boldsymbol{B}\boldsymbol{A}^{n_2} \ldots \boldsymbol{B}\boldsymbol{A}^{n_k}$, we can determine $f_W(x)$ based on the number of $\boldsymbol{B}$'s in the product. In general, $f_W(x)$ is a polynomial with integer coefficients. The degree of $f_W(x)$ is almost always the number of $\boldsymbol{B}$'s in $\boldsymbol{W}$. Below are applications of this with a small number of $\boldsymbol{B}$'s.

3.1 $\boldsymbol{f_W(x)}$ for Products with One $\boldsymbol{B}$

Recall, from Corollary 2.6,

$$\boldsymbol{B}\mathbf{A}^{\mathrm{n}} = \begin{bmatrix} x\mathrm{U}_{\mathrm{n+2}} & -\mathrm{Q}x\mathrm{U}_{\mathrm{n+1}} \\ \mathrm{U}_{\mathrm{n+1}} & -\mathrm{QU}_n \end{bmatrix}.$$

Let $\boldsymbol{W} = \boldsymbol{B}\mathbf{A}^{\mathrm{n}}$, then

$$\begin{aligned} f_W(x) &= \operatorname{tr} \boldsymbol{W} - \det \boldsymbol{W} - 1 \\ &= \operatorname{tr} \boldsymbol{W} - \det \boldsymbol{B}\,(\det \boldsymbol{A})^n - 1 \\ &= x\mathrm{U}_{\mathrm{n+2}} - \mathrm{QU}_n - Qx(Q^n) - 1 \\ &= x\mathrm{U}_{\mathrm{n+2}} - Q^{n+1}x - \mathrm{QU}_n - 1. \end{aligned}$$

Thus

$$f_W(x) = x(\mathrm{U}_{\mathrm{n+2}} - Q^{n+1}) - (\mathrm{QU}_n + 1). \tag{3.1}$$

Let us examine $f_W(x)$ for some interesting values of P and Q. We know from the introduction that $U_n = \frac{1}{3}(4^n - 1)$ when $P = 5$ and $Q = 4$. In this case,

$$f_W(x) = x(\frac{1}{3}(4^{n+2} - 1) - 4^{n+1}) - (4\left(\frac{1}{3}(4^n - 1)\right) + 1).$$

With some basic algebra, we get

$$f_W(x) = \frac{1}{3}(x - 1)(4^{n+1} - 1).$$

Just as above, there are several instances where $f_W(x)$ simplifies quite nicely. For example, when $P = 2$ and $Q = 1$. In this case, the characteristic polynomial is $x^2 - 2x + 1$, thus $x = 1$ is a repeated characteristic root. Using this, and the initials conditions, we get $U_n = n$. Therefore,

$$\begin{aligned} f_W(x) &= x(n + 2 - 1) - (n + 1) \\ &= (x - 1)(n + 1) = (x - 1)U_{n+1}. \end{aligned}$$

As expected, $x = 1$ is always a solution when $P = 2$ and $Q = 1$ just as it is when $P = 5$ and $Q = 4$. In fact, when $P = Q + 1$, not only is $x = 1$ always a root, but there is a general form for $f_W(x)$. When $P = Q + 1$, the characteristic polynomial is $x^2 - (Q + 1)x + Q$. Thus $U_n = \frac{Q^n - 1}{Q - 1}$ since the characteristic roots are $x = 1$ and $x = Q$. We can use this information to rewrite $f_W(x)$ as

$$x\left(\left(\frac{Q^{n+2} - 1}{Q - 1}\right) - Q^{n+1}\right) - \left(Q\left(\frac{Q^n - 1}{Q - 1}\right) + 1\right).$$

With some algebra, we get

$$f_W(x) = \frac{1}{Q - 1}(x - 1)(Q^{n+1} - 1) = (x - 1)U_{n+1}$$

whenever $P = Q + 1$.

Another interesting case of $f_W(x)$ occurs when $P = Q = 1$. In this case, the characteristic roots are imaginary. Therefore, it is easiest to look at the sequence which satisfies $U_n = U_{n-1} - U_{n-2}$. This sequence is $0, 1, 1, 0, -1, -1, 0, 1, 1, \ldots$ Thus we have the following table.

n	$f_W(x)$
0	-1
1	$-x - 2$
2	$-2x - 2$
3	$-2x - 1$
4	$-x$
5	0

It turns out that $\boldsymbol{A}^6 = I$. As a result, $f_{BA^n}(x) = f_{BA^{n+6}}(x)$. This means one need only consider the cases where $0 \le n \le 5$.

3.2 $\boldsymbol{f_W(x)}$ for Products with Two $\boldsymbol{B}$'s

As mentioned above, we can look at any string of $\boldsymbol{A}$'s and $\boldsymbol{B}$'s as the product of matrices of the form of $\boldsymbol{B}\mathbf{A}^{\mathrm{n}}$. For instance, the matrix representing a string of length $n+m+2$ with two $\boldsymbol{B}$'s is

$$\boldsymbol{B}\mathbf{A}^{\mathrm{n}}\boldsymbol{B}\mathbf{A}^{m} = \begin{bmatrix} x\mathrm{U}_{\mathrm{n+2}} & -\mathrm{Q}x\mathrm{U}_{\mathrm{n+1}} \\ \mathrm{U}_{\mathrm{n+1}} & -\mathrm{QU}_{n} \end{bmatrix} * \begin{bmatrix} xU_{m+2} & -QxU_{m+1} \\ U_{m+1} & -QU_{m} \end{bmatrix}$$
$$= \begin{bmatrix} x^2\mathrm{U}_{\mathrm{n+2}}\mathrm{U}_{m+2} - \mathrm{Q}x\mathrm{U}_{\mathrm{n+1}}\mathrm{U}_{m+1} & -\mathrm{Q}x^2\mathrm{U}_{\mathrm{n+2}}\mathrm{U}_{m+1} + Q^2x\mathrm{U}_{\mathrm{n+1}}\mathrm{U}_{m} \\ x\mathrm{U}_{\mathrm{n+1}}\mathrm{U}_{m+2} - \mathrm{QU}_{n}\mathrm{U}_{m+1} & -\mathrm{Q}x\mathrm{U}_{\mathrm{n+1}}\mathrm{U}_{m+1} + Q^2\mathrm{U}_{\mathrm{n}}\mathrm{U}_{m} \end{bmatrix}.$$

Here, we let $\boldsymbol{W} = \boldsymbol{B}\mathbf{A}^{\mathrm{n}}\boldsymbol{B}\mathbf{A}^{m}$ and can again easily find $f_W(x)$. Using Lemma 2.7 we get

$$f_W(x) = x^2(\mathrm{U}_{\mathrm{n+2}}\mathrm{U}_{m+2} - Q^{n+m+2}) - x(2Q\mathrm{U}_{\mathrm{n+1}}\mathrm{U}_{m+1}) + (Q^2\mathrm{U}_{\mathrm{n}}\mathrm{U}_{m} - 1). \tag{3.2}$$

Again, let us examine $f_W(x)$ for certain values of P and Q. From above, we know $U_n = n$ when $P = 2$ and $Q = 1$. Therefore,

$$f_W(x) = x^2[(n+2)(m+2) - 1] - x[2(n+1)(m+1)] + (nm - 1).$$

Also mentioned previously, we know $x = 1$ must be a solution to this. With some algebra, we get

$$f_W(x) = (\mathrm{x} - 1)[x(nm + 2n + 2m + 3) - nm + 1].$$

Once again, we can use the fact that $P = Q + 1$ to find a nice formula for $f_W(x)$. As a reminder, in this case, $U_n = \frac{Q^n - 1}{Q-1}$. With substitution, we get

$$f_W(x) = x^2\left(\left(\frac{Q^{n+2}-1}{Q-1}\right)\left(\frac{Q^{m+2}-1}{Q-1}\right) - Q^{n+m+2}\right) - x\left(2Q\left(\frac{Q^{n+1}-1}{Q-1}\right)\left(\frac{Q^{m+1}-1}{Q-1}\right)\right)$$
$$+ \left(Q^2\left(\frac{Q^{n}-1}{Q-1}\right)\left(\frac{Q^{m}-1}{Q-1}\right) - 1\right).$$

As noted above, we know $x = 1$ is a factor, therefore we can factor and simplify $f_W(x)$ to

$$\frac{1}{(Q-1)^2}(x-1)\left[x\left((2Q-1)Q^{n+m+2} - Q^{n+2} - Q^{m+2} + 1\right) - (Q^{n+m+2} - Q^{n+2} - Q^{m+2} + 2Q - 1)\right].$$

In the case with one $\boldsymbol{B}$, we saw the six possible values for $f_W(x)$ when $P = 1$ and $Q = 1$. When we increase to two $\boldsymbol{B}$'s the number of possibilities increases to 21. A sample of these cases is below.

n	m	$f_W(x)$
0	0	$-2x-1$
1	0	$-x^2-2x-1$
2	2	0
3	2	-1
4	2	$-x^2-2$
5	3	$-2x^2-1$

3.3 $\boldsymbol{f_W(x)}$ for Products with Three $\boldsymbol{B}$'s

If there are three $\boldsymbol{B}$'s, we can write $\boldsymbol{W}$ in the form $\boldsymbol{B}\mathbf{A}^{\mathrm{n}}\boldsymbol{B}\mathbf{A}^{m}\boldsymbol{B}\mathbf{A}^{\mathrm{k}}$. In this case,

$$\boldsymbol{W} = \begin{bmatrix} x^2\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2} - \mathrm{Q}x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1} & -\mathrm{Q}x^2\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+1} + Q^2x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_m \\ x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2} - \mathrm{Q}\mathrm{U}_n\mathrm{U}_{m+1} & -\mathrm{Q}x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1} + Q^2\mathrm{U}_{\mathrm{n}}\mathrm{U}_m \end{bmatrix} * \begin{bmatrix} xU_{k+2} & -QxU_{k+1} \\ U_{k+1} & -QU_k \end{bmatrix}.$$

Therefore,

$$\boldsymbol{W} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$a = x^3\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2}U_{k+2} - \mathrm{Q}x^2\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}U_{k+2} - \mathrm{Q}x^2\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+1}U_{k+1} + Q^2x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_mU_{k+1},$$
$$b = -Qx^3\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2}\mathrm{U}_{k+1} + \mathrm{Q}^2x^2\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}\mathrm{U}_{k+1} + \mathrm{Q}^2x^2\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+1}\mathrm{U}_k - Q^3x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_m\mathrm{U}_k,$$
$$c = x^2\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2}U_{k+2} - \mathrm{Q}x\mathrm{U}_n\mathrm{U}_{m+1}U_{k+2} - \mathrm{Q}x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}U_{k+1} + Q^2\mathrm{U}_{\mathrm{n}}\mathrm{U}_mU_{k+1},$$
$$d = -Qx^2\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2}U_{k+1} + \mathrm{Q}^2x\mathrm{U}_n\mathrm{U}_{m+1}U_{k+1} + \mathrm{Q}^2x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}\mathrm{U}_k - Q^3\mathrm{U}_{\mathrm{n}}\mathrm{U}_m\mathrm{U}_k.$$

This gives us

$$
\begin{aligned}
f_W(x) &= x^3\left(\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2}\mathrm{U}_{k+2} - Q^{n+m+k+3}\right) \\
&- x^2 Q(\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+1}U_{k+1} + \mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2}U_{k+1} + \mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}U_{k+2}) \\
&+ xQ^2(\mathrm{U}_{\mathrm{n}}\mathrm{U}_{m+1}U_{k+1} + \mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m}U_{k+1} + \mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}U_{k}) \\
&- (Q^3\mathrm{U}_{\mathrm{n}}\mathrm{U}_{m}U_{k} + 1).
\end{aligned} \tag{3.3}
$$

Again, let us take a look at $f_W(x)$ when $P = 2$ and $Q = 1$. Recall, in this case $U_n = n$ thus

$$
\begin{aligned}
f_W(x) = x^3\big((n+2)(m+2)(k+2) - 1\big) \\
- x^2\big((n+2)(m+1)(k+1) + (n+1)(m+2)(k+1) + (n+1)(m+1)(k+2)\big) \\
+ x\big((n)(m+1)(k+1) + (n+1)(m)(k+1) + (n+1)(m+1)(k)\big) - (nmk+1).
\end{aligned}
$$

As before, $x = 1$ must be a root, therefore

$$
\begin{aligned}
f_W(x) = (x-1)(x^2(nmk + 2mn + 2mk + 2nk + 4n + 4m + 4k + 7) \\
- x(2nmk + 2mn + 2mk + 2nk + n + m + k - 1) + nmk + 1).
\end{aligned}
$$

3.4 $\boldsymbol{f_W(x)}$ for Products with Four $\boldsymbol{B}$'s

As you can see, we are able to continue this process for any number of $\boldsymbol{B}$'s. As mentioned previously, the degree of $f_W(x)$ is usually the number of $\boldsymbol{B}$'s in $\boldsymbol{W}$. Consequently, the larger the number of $\boldsymbol{B}$'s in the product, the more complicated $f_W(x)$ will be. For example, when $\boldsymbol{W} = \boldsymbol{B}\mathbf{A}^{\mathrm{n}}\boldsymbol{B}\mathbf{A}^{m}\boldsymbol{B}\mathbf{A}^{\mathrm{k}}\boldsymbol{B}\mathbf{A}^{\mathrm{j}}$ we have

$$
\begin{aligned}
\operatorname{tr}\boldsymbol{W} = x^4\big(\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2}U_{k+2}U_{j+2}\big) \\
- x^3 Q\big(\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}U_{k+2}U_{j+2} + \mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+1}U_{k+1}U_{j+2} + \mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2}\mathrm{U}_{k+1}U_{j+1} \\
+ \mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2}U_{k+2}U_{j+1}\big) \\
+ x^2 Q^2\big(\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m}U_{k+1}U_{j+2} + 2\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}\mathrm{U}_{k+1}U_{j+1} + \mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+1}\mathrm{U}_{k}U_{j+1} \\
+ \mathrm{U}_{n}\mathrm{U}_{m+1}U_{k+2}U_{j+1} + \mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2}U_{k+1}U_{j}\big) \\
- xQ^3\big(\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m}\mathrm{U}_{k}U_{j+1} + \mathrm{U}_{\mathrm{n}}\mathrm{U}_{m}U_{k+1}U_{j+1} + \mathrm{U}_{n}\mathrm{U}_{m+1}U_{k+1}U_{j} + \mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}\mathrm{U}_{k}U_{j}\big) \\
+ Q^4\mathrm{U}_{n}\mathrm{U}_{m}\mathrm{U}_{k}U_{j}
\end{aligned}
$$

and

$$
\det\boldsymbol{W} = Q^{n+m+k+j+4}x^4.
$$

Therefore,

$$
\begin{aligned}
f_W(x) = x^4&\left(\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2}U_{k+2}U_{j+2} - Q^{n+m+k+j+4}\right) \\
&- x^3Q\left(\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}U_{k+2}U_{j+2} + \mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+1}U_{k+1}U_{j+2}\right. \\
&\left. + \mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2}\mathrm{U}_{k+1}U_{j+1} + \mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2}U_{k+2}U_{j+1}\right) \\
&+ x^2Q^2\left(\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m}U_{k+1}U_{j+2} + 2\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}\mathrm{U}_{k+1}U_{j+1}\right. \\
&\left. + \mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+1}\mathrm{U}_{k}U_{j+1} + \mathrm{U}_{n}\mathrm{U}_{m+1}U_{k+2}U_{j+1} + \mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2}U_{k+1}U_{j}\right) \\
&- xQ^3\left(\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m}\mathrm{U}_{k}U_{j+1} + \mathrm{U}_{\mathrm{n}}\mathrm{U}_{m}U_{k+1}U_{j+1} + \mathrm{U}_{n}\mathrm{U}_{m+1}U_{k+1}U_{j}\right. \\
&\left. + \mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}\mathrm{U}_{k}U_{j}\right) + (Q^4\mathrm{U}_{n}\mathrm{U}_{m}\mathrm{U}_{k}U_{j} - 1).
\end{aligned}
\tag{3.4}
$$

Just as above, we examine $f_W(x)$ for a case where we know $x = 1$ will occur. Using the fact that $U_n = n$ when $P = 2$ and $Q = 1$, we can rewrite $f_W(x)$ as

$$
\begin{aligned}
x^4&((\mathrm{n}+2)(m+2)(k+2)(j+2) - 1) \\
&- x^3\big((\mathrm{n}+1)(m+1)(k+2)(j+2) + (\mathrm{n}+2)(m+1)(k+1)(j+2) \\
&+ (\mathrm{n}+2)(m+2)(k+1)(j+1) + (\mathrm{n}+1)(m+2)(k+2)(j+1)\big) + x^2((\mathrm{n} \\
&+ 1)(m)(k+1)(j+2) + 2(\mathrm{n}+1)(m+1)(k+1)(j+1) + (\mathrm{n}+2)(m+1)(k)(j \\
&+ 1) + (n)(m+1)(k+2)(j+1) + (\mathrm{n}+1)(m+2)(k+1)(j)\,) \\
&- x((\mathrm{n}+1)(m)(k)(j+1) + (\mathrm{n})(m)(k+1)(j+1) + (n)(m+1)(k+1)(j) + (\mathrm{n} \\
&+ 1)(m+1)(k)(j)) + (nmkj - 1).
\end{aligned}
$$

As mentioned previously, we know $x = 1$ is a root in this case. Therefore, we can factor $f_W(x)$ in to

$$
\begin{aligned}
(x-1)(x^3&(nmkj + 2nmk + 2nmj + 2njk + 2mkj + 4nm + 4nk + 4nj + 4mk + 4mj + 4kj + 8n \\
&+ 8m + 8k + 8j + 15) \\
&- x^2(3nmkj + 4nmk + 4nmj + 4nkj + 4mkj + 5nm + 4nk + 5nj + 5mk + 4mj \\
&+ 5kj + 4n + 4m + 4k + 4j + 1) \\
&+ x(3nmkj + 2nmk + 2nmj + 2nkj + 2mkj + nm + nj + mk + kj + 1) - nmkj \\
&+ 1).
\end{aligned}
$$

4 Results

Using the functions found in Chapter 3, we can now look for periodic solutions associated with $\boldsymbol{W}$ where $\boldsymbol{W}$ has a specific number of $\boldsymbol{B}$'s. Below we address the x value and initial conditions that we expect to lead to periodic solutions. Recall, as mentioned previously, the initial conditions which allow for periodic behavior can be found from the eigenvector. In general, if $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has 1 as an eigenvalue, then $\begin{bmatrix} 1-d \\ c \end{bmatrix}$ is an eigenvector, unless $c = 0$ and $d = 1$. We will use this to find the initial conditions for cases with a small number of $\boldsymbol{B}$'s below. In all of these cases, Mathematica was used to solve for x given a specific P and Q and then calculate the initial conditions as defined below. Finally, as mentioned previously, Theorem 2.2 is necessary but not sufficient to achieve periodic behavior. Therefore, results were tested to verify that they did indeed lead to periodic solutions. The results of this search are also included below.

As mentioned previously, there are three commonly occurring x values which must be addressed. These three values are $x = 1$, $x = -1$, and $x = 0$. In Chapter 3 we addressed what we can say about the occurrence of these values, here we will address the consequence of their occurrence. First, when $x = 1$, the solution is already periodic. In this instance, (1.6) would simply become (1.1). These were the linear cases addressed in Section 2.2. Similar to $x = 1$, when $x = -1$, the recurrence would merely be $a_n = -Pa_{n-1} + Qa_{n-2}$. As mentioned in Section 2.2, this would have a periodic solution with a period half the length of $a_n = Pa_{n-1} + Qa_{n-2}$. Again, these were the linear cases addressed in Section 2.2. In the case where $x = 0$, we would receive the trivial solution $a_0, a_1, 0, 0, 0, \ldots$ Therefore, these cases were uninteresting to our overall search and were therefore disregarded. Also, it seems likely that the only time periodic behavior is possible is when $|x| < 1$. While we did not prove this, there were no cases in our research in which a periodic solution had $|x| > 1$.

4.1 Products with One $\boldsymbol{B}$

Recall, we are interested in the x values for which $f_W(x) = 0$. Therefore, we solve

$$x(\mathrm{U_{n+2}} - Q^{n+1}) - (\mathrm{QU}_n + 1) = 0$$

for x. Thus for fixed P and Q, we need $x = \frac{1+QU_n}{U_{n+2}-Q^{n+1}}$ to allow for periodic behavior. Also, as discussed above, since

$$\boldsymbol{W} = \begin{bmatrix} x\mathrm{U_{n+2}} & -\mathrm{Q}x\mathrm{U_{n+1}} \\ \mathrm{U_{n+1}} & -\mathrm{QU}_n \end{bmatrix},$$

an eigenvector with eigenvalue 1 is

$$\begin{bmatrix} \mathrm{QU}_n + 1 \\ \mathrm{U_{n+1}} \end{bmatrix}.$$

Thus when $x = \frac{1+QU_n}{U_{n+2}-Q^{n+1}}$, $a_0 = \mathrm{U_{n+1}}$ and $a_1 = \mathrm{QU}_n + 1$ it is possible to achieve periodic solutions. Note that in order for x to be defined we need $U_{n+2} \neq Q^{n+1}$. In our search, x was rarely undefined. There were two scenarios where this occurred. The first is when $P = Q = 1$. In this case, x is undefined when $n = 0$ and $n = 5$, as can be seen in the table in Section 3.1. Recall, as mentioned previously, we only need to consider $0 \leq n \leq 5$ in this case. The second problematic case occurs when $P = Q$ and $n = 0$. This causes $U_{n+2} = Q^{n+1}$ and thus x is undefined. We ignored the cases where x was undefined.

As a reminder, in this search, we restrict P and Q such that $1 \leq P \leq 20$ and $-20 \leq Q \leq 20$ and $Q \neq 0$. Furthermore, as mentioned before, the $P = Q = 1$ case was frequently problematic. Therefore, from now on we will disregard this case. Often, we considered only products with $0 \leq n \leq 30$. However, occasionally that limit was expanded to search for possible patterns. Given our restrictions above, there are 24,764 cases to be considered. Ignoring the cases where $x = 1$, $x = -1$ or $x = 0$, there are 23,836 cases remaining. Of these, only 48 do not lead to periodic solutions. Below is a table outlining these cases.

$\boldsymbol{P, Q}$	$\boldsymbol{n}$	$\boldsymbol{a_0, a_1}$	$\boldsymbol{x}$
P, P	1	$P, P+1$	$-(P+1)/P$
$P, -P\ (P \neq 1)$	1	$P, -(P-1)$	$-(P-1)/P$
$3, -3$	5	$81, -64$	$-8/27$

$4, 6$	2	$2, 5$	$-1/8$
$6, 3$	3	$9, 5$	$1/9$
$8, -8$	3	$128, -115$	$-23/64$
$9, 11$	5	$24, 35$	$-1/35$
$10, 10$	5	$1000, 1127$	$-161/1000$
$10, 12$	2	$8, 11$	$-1/8$
$11, -11$	5	$1331, -1228$	$-307/1331$
$12, 14$	2	$10, 13$	$-1/8$
$18, 20$	2	$16, 19$	$-1/8$

Table 4.1

Consider the entries in Table 4.1 where $n = 2$. We noticed that in all of these cases $x = -\frac{1}{8}$. As a result, we believe when $P \equiv 2(mod\ 8)$ or $P \equiv 4(mod\ 8)$, $Q = P + 2$ and $n = 2$ there will not be a periodic solution. An explanation for this is given in Section 5.1. There is one case where there is a solution which is actually periodic, yet not as expected. When $P = 2$, $Q = -2$ and $n = 1$ our method gives us $x = -\frac{1}{2}$ and suggests that starting with $a_0 = 2$ and $a_1 = -1$ should lead to a periodic solution with period two. Instead, the result is actually $2, -1, -1, 2, -1, \ldots$ which is periodic, but with period three rather than the expected period of two.

4.2 Products with Two $\boldsymbol{B}$'s

Recall, generally, the degree of $f_W(x)$ is the number of $\boldsymbol{B}$'s in $\boldsymbol{W}$. Since $\boldsymbol{W} = \boldsymbol{B}\boldsymbol{A}^n\boldsymbol{B}\boldsymbol{A}^m$, our formula for $f_W(x)$ is now a quadratic. Therefore, instead of finding a specific formula for x, Mathematica was used to solve for x given a specific P, Q, m, and n. Just as in the One $\boldsymbol{B}$ case, we use the eigenvector to determine the initial conditions. Recall from Section 3.2, that

$$\boldsymbol{W} = \begin{bmatrix} x^2\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2} - \mathrm{Q}x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1} & -\mathrm{Q}x^2\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+1} + Q^2x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_m \\ x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2} - \mathrm{Q}\mathrm{U}_n\mathrm{U}_{m+1} & -\mathrm{Q}x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1} + Q^2\mathrm{U}_{\mathrm{n}}\mathrm{U}_m \end{bmatrix}.$$

From the introduction to this chapter, we know that

$$\begin{bmatrix} \mathrm{Q}x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1} - Q^2\mathrm{U}_{\mathrm{n}}\mathrm{U}_m + 1 \\ x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2} - \mathrm{Q}\mathrm{U}_n\mathrm{U}_{m+1} \end{bmatrix}$$

is an eigenvector with eigenvalue 1. Unlike the One $\boldsymbol{B}$ case, we cannot simply take $a_0 = x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2} - \mathrm{Q}\mathrm{U}_n\mathrm{U}_{m+1}$ and $a_1 = \mathrm{Q}x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1} - Q^2\mathrm{U}_{\mathrm{n}}\mathrm{U}_m + 1$. This is because x is a rational number, yet we want integers for our initial conditions. Consequently, Mathematica was used to determine the eigenvector and properly scale it so that the initial conditions are always integers.

As mentioned in Chapter 3, there are times where $x = 1$, $x = -1$ or $x = 0$ can occur. However, unlike the single $\boldsymbol{B}$ case, when there are two $\boldsymbol{B}$'s there is an additional solution for x in all of these cases since $f_W(x)$ is a quadratic. To determine what the remaining x value is for each case, consider the generic polynomial $ax^2 + bx + c$. Let r_1 and r_2 denote the roots of this polynomial. From Vieta's formulas we can relate these roots and the coefficients of the polynomial. Therefore, we know $r_1 + r_2 = -\frac{b}{a}$. Since we know one of the roots, we subtract r_2 to get $r_1 = -\frac{b}{a} - r_2$. Thus when $x = 1$, the remaining solution is

$$x = \frac{2Q\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}}{\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2} - Q^{n+m+2}} - 1.$$

Similarly, when $x = -1$, the other solution is

$$x = \frac{2Q\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}}{\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2} - Q^{n+m+2}} + 1.$$

Finally, when $x = 0$, the remaining solution is

$$x = \frac{2Q\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}}{\mathrm{U}_{\mathrm{n}+2}\mathrm{U}_{m+2} - Q^{n+m+2}}.$$

Another case where we know the quadratic will factor is when $n = m$, with $\boldsymbol{W} = (\boldsymbol{BA}^n)^2$. Recall if c_1 and c_2 are eigenvalues of $\boldsymbol{W}$ then c_1^m and c_2^m are eigenvalues of $\boldsymbol{W}^m$. Therefore, if 1 is an eigenvalue of $\boldsymbol{BA}^n$ then 1 is also an eigenvalue of $(\boldsymbol{BA}^n)^2$. This means that the x value from the One $\boldsymbol{B}$ case will be one of the solutions to $f_W(x)$ in the Two $\boldsymbol{B}$ case. Therefore, we are able to find an explicit formula for the second root. As mentioned above, we are interested in the cases where $n = m$, thus

$$f_W(x) = x^2\left({\mathrm{U}_{\mathrm{n}+2}}^2 - Q^{2n+2}\right) - x\left(2Q{\mathrm{U}_{\mathrm{n}+1}}^2\right) + \left(Q^2{\mathrm{U}_{\mathrm{n}}}^2 - 1\right).$$

We can use Lemma 2.7 to rewrite $f_W(x)$ as

$$x^2\left({\mathrm{U}_{\mathrm{n}+2}}^2 - Q^{2n+2}\right) - x\left(2Q(Q^n + U_{n+2}U_n)\right) + \left(Q^2{\mathrm{U}_{\mathrm{n}}}^2 - 1\right).$$

From here, we can easily factor $f_W(x)$. Thus we have

$$f_W(x) = \big((U_{n+2} - Q^{n+1})x - (QU_n + 1)\big)\big((U_{n+2} + Q^{n+1})x - (QU_n - 1)\big).$$

As a reminder, the solution from the One $\boldsymbol{B}$ case is $x = \frac{QU_n+1}{U_{n+2}-Q^{n+1}}$. Therefore, we have

$$x = \frac{QU_n - 1}{U_{n+2} + Q^{n+1}}$$

as the second root when $\boldsymbol{W} = (\boldsymbol{BA}^n)^2$. In this case, the initial conditions are $a_0 = U_{n+1}$ and $a_1 = QU_n - 1$.

Since we know the product $(\boldsymbol{BA}^n)^2$ has two clearly defined x values, we expect there to frequently be two periodic solutions when $n = m$. We found some cases where there is only one solution. To begin with, all of the cases in Table 4.1, that do not lead to periodic behavior in the One $\boldsymbol{B}$ case, attribute to a missing solution in the Two $\boldsymbol{B}$ case. Additionally, when $n = 1$, P is arbitrary and $Q = 1$ or $Q = -1$, $x = 0$ is one of the roots, thus there is only one x value used to look for a periodic solution. Additionally, two new patterns arose. Finally, there was one outlier. These cases are listed in the table below.

P	Q	$\boldsymbol{W}$	a_0, a_1	x
2	1	$(\boldsymbol{BA}^{4k+1})^2, k \in \mathbb{Z}$	$4k+2, 4k$	$\frac{k}{k+1}$
$4(mod\ 8)$ or $6(mod\ 8)$	$-P+2$	$(\boldsymbol{BA}^2)^2$	$P+2, 1-P$	$-\frac{1}{8}$
5	-7	$(\boldsymbol{BA}^3)^2$	$13, -15$	$-\frac{1}{16}$

By Corollary 2.4, $\boldsymbol{W} = \boldsymbol{B}\mathrm{A}^{\mathrm{n}}\boldsymbol{B}\mathrm{A}^m$ and $\boldsymbol{W} = \boldsymbol{B}\mathrm{A}^m\boldsymbol{B}\mathrm{A}^{\mathrm{n}}$ have the same function $f_W(x)$. Therefore, without loss of generality, we can restrict our search to cases with $n \geq m$. Furthermore, as mentioned previously, the $P = Q = 1$ case was frequently problematic. Therefore, as before, we will disregard this case. There are 396,304 possible combinations of $P, Q, m,$ and n given our restrictions. Of these, $f_W(x)$ does not factor in 356,988 cases. Of the 39,316 remaining cases which factor, there are 63,618 x values with $x \neq -1, 0, 1$. These x values lead to 62,975 periodic solutions. There are 643 cases where a rational x value does not lead to periodic behavior.

As mentioned earlier, we can characterize when $x = 1$ is a root. Previously, we determined that $x = 1$ occurs when $P = Q + 1$ and $P = -Q - 1$. We know that in these cases there will be a second solution. In fact, there are 14,367 occurrences of $x = -1, 0, 1$. Of these, only 30 of the subsequent x values do not lead to periodic behavior. This only appears to happen when $P = 2$ and $Q = 1$. Of these cases, there appears to be a pattern, however we were unable to decipher exactly what it is. However, after finding $f_W(x)$ in Section 3.2, specifically when $P = 2$ and $Q = 1$, we are able to state exactly what the second x value is. In this case, our second root is $x = \frac{nm-1}{nm+2n+2m+3}$. For this x value, the initial conditions become $a_0 = mn + 2m + 1$ and $a_1 = mn - 1$.

4.3 Products with Three $\boldsymbol{B}$'s

Just as before, we need to solve for x and the initial conditions. Since $f_W(x)$ is cubic, we again used Mathematica to solve for x. Additionally, we know that

$$\begin{bmatrix} Qx^2\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2}U_{k+1} - \mathrm{Q}^2x\mathrm{U}_n\mathrm{U}_{m+1}U_{k+1} - \mathrm{Q}^2x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}\mathrm{U}_k + Q^3\mathrm{U}_{\mathrm{n}}\mathrm{U}_m\mathrm{U}_k + 1 \\ x^2\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+2}U_{k+2} - \mathrm{Q}x\mathrm{U}_n\mathrm{U}_{m+1}U_{k+2} - \mathrm{Q}x\mathrm{U}_{\mathrm{n}+1}\mathrm{U}_{m+1}U_{k+1} + Q^2\mathrm{U}_{\mathrm{n}}\mathrm{U}_m U_{k+1} \end{bmatrix}$$

is an eigenvector, with eigenvalue 1, of $\boldsymbol{W}$ from Section 3.3. Just as in the Two $\boldsymbol{B}$ case, we cannot use these values directly as our initial conditions. Thus, once again, we used Mathematica to find and scale the eigenvector appropriately.

Just as in the Two $\boldsymbol{B}$ case, we can limit the exponents on $\boldsymbol{A}$ to avoid double counting cyclic permutations. In this case, we want $n \geq m$ and $n \geq k$. With these restrictions, we have 8,322,384 possible combinations of P, Q, n, m and k. Of these, 7,985,175 cases did not factor and thus did not have any rational x values. Of the remaining combinations, 329,689 cases had one rational x value, 22 cases had two distinct rational x values and 7,498 cases had three x values.

When looking at products of the form $(\boldsymbol{BA}^n)^3$, we expect to have at least one periodic solution. This comes from the x value and periodic solution from the One $\boldsymbol{B}$ case. That is, if $\boldsymbol{v}$ is an eigenvector for $\boldsymbol{BA}^n$, then $\boldsymbol{v}$ is also an eigenvector for $(\boldsymbol{BA}^n)^3$. In most cases, this was the only periodic solution. In a few cases, solutions one might expect to find were missing. For example, when $\boldsymbol{W} = (\boldsymbol{BA})^3$, the cases

$P = Q$, $-P = Q$, and $Q = -1$ did not have any periodic solutions. This is because in the One $\boldsymbol{B}$ case these cases also did not have a periodic solution. Table 4.1 outlines the first two cases, and when P is arbitrary and $Q = -1$ the resulting x value is zero. This was frequently the problem with missing solutions. For instance, when $P = -Q - 1$ we saw in Chapter 3 that $x = -1$ will be a solution. As a result, the product $(\boldsymbol{B}\boldsymbol{A}^n)^3$ does not have periodic solutions when n is odd in this case. Similarly, when $P = Q + 1$ we know $x = 1$ is a solution. Consequently, there are no periodic solutions in this case, for $P \geq 3$.

4.4 Products with Four $\boldsymbol{B}$'s

As mentioned before, the matrix $\boldsymbol{W}$, and thus the function $f_W(x)$, becomes increasingly more complicated when the number of $\boldsymbol{B}$'s in the product increases. Therefore, for products with Four $\boldsymbol{B}$'s, we will not list the matrix $\boldsymbol{W}$ or an eigenvector with eigenvalue 1. Recall, we are looking for the rational roots of (3.4). These x values have the potential to lead to periodic solutions. Once again, Mathematica was used to find the rational x values for which $f_W(x) = 0$. Additionally, Mathematica was used to find the eigensystem and scale the eigenvector appropriately for the initial conditions.

Similar to previous cases, we restricted the exponents of $\boldsymbol{A}$ to avoid double counting cyclic permutations. When $\boldsymbol{W} = \boldsymbol{B}\mathbf{A}^{\mathrm{n}}\boldsymbol{B}\mathbf{A}^{m}\boldsymbol{B}\mathbf{A}^{\mathrm{k}}\boldsymbol{B}\mathbf{A}^{\mathrm{j}}$ our limits are $n \geq m$, $n \geq k$ and $n \geq j$. As a reminder, the case where $P = Q = 1$ was also disregarded. This gives 42,635,439 possible combinations of P, Q, n, m, k and j. Of these, 41,064,688 cases had no rational x values. Of the remaining cases that factored, 1,515,213 cases had one rational x value, 55,488 cases had two distinct rational x values, 49 cases had three distinct rational x values and 1 case had four distinct rational x values.

The only case where there were four rational x values occurred when $P = 20$, $Q = -5$ and $\boldsymbol{W} = (\boldsymbol{B}^2\boldsymbol{A})^2$. In this case,

$$\begin{aligned} f_W(x) &= 65{,}594{,}375x^4 + 3{,}240{,}000x^3 + 40{,}250x^2 - 1 \\ &= (235x - 1)(145x + 1)(55x + 1)(35x + 1) \end{aligned}$$

so the rational x values we get are $x=-\frac{1}{35}$, $x=-\frac{1}{55}$, $x=-\frac{1}{145}$, $x=\frac{1}{235}$. The values $x=-\frac{1}{35}$ and $x=\frac{1}{235}$ were the two periodic solutions from the Two $\boldsymbol{B}$ case. The remaining values, $x=-\frac{1}{55}$ and $x=-\frac{1}{145}$ lead to two new periodic solutions in this Four $\boldsymbol{B}$ case.

For the 49 cases with three rational x values, we know $f_W(x)$ must factor into linear terms. In fact, in all of these cases $f_W(x)$ factored into the quadratic from the Two $\boldsymbol{B}$ case and a quadratic which was a perfect square. In other words, all of these cases had the two x values from the Two $\boldsymbol{B}$ case and then a new x value, of multiplicity two, from the Four $\boldsymbol{B}$ case. For all 49 cases, the new x value led to an additional periodic solution.

The majority of the time $f_W(x)$ factored into the quadratic from the Two $\boldsymbol{B}$ case and another quadratic which did not factor. For example, when $P=3$ and $Q=-1$, the matrix $(\boldsymbol{B}\mathbf{A}^2)^2$ yields the function

$$\begin{aligned} f_W(x) &= 1088x^2+200x+8 \\ &= 8(8x+1)(17x+1) \end{aligned}$$

Thus $x=-\frac{1}{8}$ and $x=-\frac{1}{17}$ were the rational roots which allowed for the possibility of periodic behavior. In the Four $\boldsymbol{B}$ case, $(\boldsymbol{B}\mathbf{A}^2)^4$,

$$\begin{aligned} f_W(x) &= 1185920x^4+435600x^3+59600x^2+3600x+80 \\ &= 80(109x^2+20x+1)(17x+1)(8x+1). \end{aligned}$$

In other words, no new periodic solutions were found in this Four $\boldsymbol{B}$ case.

5 Conjectures

As mentioned before, there are several scenarios where patterns seem to arise in our data. Below we will address the patterns we believe exist. Although we believe these to be actual patterns that will be true for all cases, this has not been proven. Proving that these are indeed infinite families is one of the suggestions for future work. Other ideas for future work are also given below.

5.1 Patterns for Products with One $\boldsymbol{B}$

As most of the solutions in the One $\boldsymbol{B}$ case were periodic, we outline patterns where periodic solutions do not appear to exist. As you can see from Table 4.1, there were two patterns we identified. We found that when $P = Q$ and $n = 1$ the result is probably not periodic. Also, when $Q = -P$ and $n = 1$ we did not find any periodic solutions. The final suspected pattern was also noted in Section 4.1. We believe when $P \equiv 2(mod\ 8)$ or $P \equiv 4(mod\ 8)$, $Q = P + 2$ and $n = 2$ there will not be a periodic solution. To check whether this holds, we expanded our restrictions on P and Q. In this case, we checked $-100 \leq Q \leq 100$. The results were consistent with the patterns we expected. In fact, if we look at the general case $Q = P + 2$, we see why this happens. When P is arbitrary and $Q = P + 2$, since $n = 2$ is quite small, we can easily find $f_W(x)$ for this case. Given these circumstances, $f_W(x) = (P+1)^2(-8x-1)$. Therefore, when P is arbitrary, $Q = P + 2$ and $n = 2$, $x = -\frac{1}{8}$ is always the solution. Note, in this case,

$$\boldsymbol{BA}^2 = \begin{bmatrix} -\frac{1}{8}(P^3 - 2P^2 - 4P) & \frac{1}{8}(P-2)(P+2)(P+1) \\ (P-2)(P+1) & -P(P+2) \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 1 - \big(-P(P+2)\big) \\ (P-2)(P+1) \end{bmatrix} = \begin{bmatrix} (P+1)^2 \\ (P-2)(P+1) \end{bmatrix}$$

is an eigenvector with eigenvalue 1. When scaling the eigenvector to find the initial conditions, we receive $a_0 = P - 2$ and $a_1 = P + 1$. Since $n = 2$, we expect there to be two terms before we can multiply by x to achieve periodic behavior as expected. If we look at the sequence resulting from the initial conditions, we have $P - 2, P + 1, P + 4, P - 2, -8(P + 1)$. In order to multiply by x in the appropriate place, we need $P + 4$ and $P - 2$ to not be divisible by 8. Otherwise, the multiplication will occur in the wrong place and probably cause the sequence to not be periodic. This explains why we did not find a periodic solution when $P \equiv 2(mod\ 8)$ or $P \equiv 4(mod\ 8)$. There is some evidence that these three families, and the six exceptions in Table 4.1, are the only cases without periodic solutions.

5.2 Patterns for Products with Two $\boldsymbol{B}$'s

Unlike the One $\boldsymbol{B}$ case, here we characterize patterns that lead to periodic solutions. For instance, when $Q = P - 1$, for $P \geq 3$, all 496 possible combinations of m and n result in periodic behavior. As a reminder, this is one of the cases where $x = 1$ is a root. In these cases, we found that the subsequent x value always led to a periodic solution. Similarly, when $Q = -P - 1$ the solution will be periodic when m and n have matching parity.

As mentioned previously, there were some cases where $(\boldsymbol{BA}^n)^2$ would not lead to periodic behavior. One of these cases, as mentioned in Section 4.2, was when $P \equiv 4(mod\ 8)$ or $P \equiv 6(mod\ 8)$, $Q = -P + 2$ and $n = 2$. This pattern is quite similar to the one explained in Section 5.1. When determining our sequence, we get $P + 2, 1 - P, P - 4, -P - 2, 8(1 - P), \ldots$ As you can see, whenever $P \equiv 4(mod\ 8)$ or $P \equiv 6(mod\ 8)$ we end up multiplying by $-\frac{1}{8}$ before the expected point. This causes the solution to not be periodic.

As mentioned previously, the square of the One $\boldsymbol{B}$ case leads to a second solution. In most cases this second x value leads to a periodic solution. All of the following cases led to periodic solutions when $\boldsymbol{W} = (\boldsymbol{BA}^n)^2$ unless noted in Section 4.2. In addition to the square case, some combinations of P and Q have an additional pattern which is addressed below.

P	Q	$\boldsymbol{W}$	x value	a_0, a_1
arbitrary	-1	$\boldsymbol{BA}^{n+2}\boldsymbol{BA}^n$ when n is odd	$\frac{-U_{n-1}}{U_{n+1}}$	$U_{n+1}, U_n - 1$
$k^2 + 1, k \in \mathbb{Z}$	1	$\boldsymbol{BA}^{n+1}\boldsymbol{BA}^n$ for all n	undetermined	

arbitrary	P^2	$\boldsymbol{B}\boldsymbol{A}^{n+6k}\boldsymbol{B}\boldsymbol{A}^n, k \in \mathbb{Z}$	undetermined
even	$\frac{1}{2}P^2$	$\boldsymbol{B}\boldsymbol{A}^{n+8k}\boldsymbol{B}\boldsymbol{A}^n, k \in \mathbb{Z}$	undetermined
divisible by 3	$\frac{1}{3}P^2$	$\boldsymbol{B}\boldsymbol{A}^{n+12k}\boldsymbol{B}\boldsymbol{A}^n, k \in \mathbb{Z}$	undetermined

When P and Q are defined as above, the corresponding product appears to lead to periodic behavior. In almost all of these cases, we checked values for P and Q greater than 20 to verify what we suspected to be an infinite family. While this doesn't prove it is indeed a family, it makes us slightly more confident in our assumption.

5.3 Patterns for Products with Three $\boldsymbol{B}$'s

Similar to the Two $\boldsymbol{B}$ case, below are conditions we believe lead to periodic solutions. Again, these are patterns in addition to the result from the One $\boldsymbol{B}$ case. In other words, when $\boldsymbol{W} = (\boldsymbol{B}\boldsymbol{A}^n)^3$, we expect there to be one periodic solution which results from the periodic solution of $\boldsymbol{B}\boldsymbol{A}^n$. Any cases where this is not true were addressed in Section 4.3.

P	Q	$\boldsymbol{W}$	$\boldsymbol{x}$ value
arbitrary	-1	$\boldsymbol{B}\boldsymbol{A}^{2n+3}\boldsymbol{B}\boldsymbol{A}^{n+2}\boldsymbol{B}\boldsymbol{A}^n$	$\frac{-U_{n+1}}{U_{n+3}}$
arbitrary	1	$\boldsymbol{B}\boldsymbol{A}^{2n+3}\boldsymbol{B}\boldsymbol{A}^{n+2}\boldsymbol{B}\boldsymbol{A}^n$	
arbitrary	P^2	undetermined	
even	$\frac{1}{2}P^2$	undetermined	
divisible by 3	$\frac{1}{3}P^2$	undetermined	

Unfortunately, these patterns were harder to determine than cases with a smaller number of $\boldsymbol{B}$'s. It is clear that the same combinations of n, m and k appear for any given pattern of P and Q, however the pattern that n, m and k follows is not easily decipherable. Therefore, we suspect these are infinite families, but have not been able to determine exactly when we suspect periodic solutions to occur.

5.4 Patterns for Products with Four $\boldsymbol{B}$'s

As mentioned previously, there were several occurrences where $f_W(x)$ factored into the quadratic from the Two $\boldsymbol{B}$ case and another quadratic which did not factor. In fact, it appears as though most products of the form $(\boldsymbol{BA}^n\boldsymbol{BA}^m)^2$, including those where $n = m$, have only periodic solutions from the Two $\boldsymbol{B}$ case. For instance, when $Q = P - 1$, for $P \geq 3$, the only periodic solutions found were of the form $(\boldsymbol{BA}^n\boldsymbol{BA}^m)^2$. These are the solutions addressed in Section 5.2. Additionally, when $Q = -P - 1$, the only periodic solutions were from the similar pattern in Section 5.2. In other words, there were frequently no new periodic solutions in the Four $\boldsymbol{B}$ case.

There were 50 cases where $f_W(x)$ factored in to all linear terms. The results of these cases were addressed in Section 4.4. Excluding the single case with four rational x values, most of these cases appear to fall in to patterns. We suspect $f_W(x)$ will factor in to linear terms for the patterns below.

P	Q	$\boldsymbol{W}$
2	1	$(\boldsymbol{BA}^{n+2}\boldsymbol{BA}^n)^2$
4	-1	$(\boldsymbol{BA}^{n+2}\boldsymbol{BA}^n)^2$ where n is even
10	1	$(\boldsymbol{BA}^{n+2}\boldsymbol{BA}^n)^2$

In addition to the patterns above, there was the case where $P = 4$, $Q = -2$ and $\boldsymbol{W} = (\boldsymbol{BAB})^2$ which had three distinct rational x values. This case did not appear to fall into any pattern.

The remaining new periodic solutions found appear to fall into patterns similar to those from cases with a smaller number of $\boldsymbol{B}$'s. These suspected patterns are outlined below.

P	Q	$\boldsymbol{W}$
arbitrary	-1	$\boldsymbol{BA}^{2n+3}\boldsymbol{BA}^{2n+3}\boldsymbol{BA}^{n+2}\boldsymbol{BA}^n$
arbitrary	P^2	undetermined
even	$\frac{1}{2}P^2$	undetermined
divisible by 3	$\frac{1}{3}P^2$	undetermined

The last three cases included periodic solutions in addition to the squares of the two $\boldsymbol{B}$ patterns. Unfortunately, just like the Three $\boldsymbol{B}$ case, it was too difficult to determine the matrices which led to periodic solutions.

6 Future Work

There are several ways in which this project might be extended. To begin with, there is simple expansion. In this project, we limited P and Q to relatively small intervals. One could consider many more cases of P and Q. This might lead to a realization of patterns which we were unable to see. In that respect, another logical step is to prove that what we suspect are infinite families are indeed just that. Furthermore, we only looked at cases with a small number of B's, one could consider products with a larger number of B's. Additionally, in this project we looked at a nonlinear variation on second order homogeneous recurrence relations. One could consider nonhomogeneous systems next.

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A Mathematica Code

Here we give the Mathematica code used for this project. The code below was used to generate a list of the U_n sequence for a specific P and Q, which was needed to calculate $f_W(x)$. A similar list is generated and stored for the a_n sequence. Furthermore, the function isPeriodic will determine whether or not a sequence is periodic. It uses the fact that we can determine the expected length of the period based on the number of matrices in the product being examined. This code, PeriodicFunctions.nb, was run as an initialization for some of the programs in the following sections.

PeriodicFunctions.nb

```
Usize=200;
Asize=200;
Areset[a1_,a2_]:=Module[{},
   A=Table[0,{i,1,Asize}];
   A[[1]]=a1;A[[2]]=a2;
   Alength=2;
   ];
Ureset[]:=Module[{},
   U=Table[0,{i,1,Usize}];
   U[[1]]=0;U[[2]]=1;
   Ulength=2;
   ];
ExpandU[index_]:=Module[{i},
   For[i=Ulength+1,i≤ index,i++,
     Ulength++;
     U[[Ulength]]=p*U[[Ulength-1]]-q*U[[Ulength-2]];
     ];
   ];
ExpandA[index_]:=Module[{i},
   For[i=Alength+1,i≤ index,i++,
     Alength++;
     A[[Alength]]=Piecewise[{{x*(p*A[[Alength-1]]-q*A[[Alength-
2]]),IntegerQ[x*(p*A[[Alength-1]]-q*A[[Alength-2]])]}},p*A[[Alength-1]]-q*A[[Alength-
2]]]];
   ];
u[m_]:=Module[{Uindex=m+1},
   If[Uindex>Ulength,ExpandU[Uindex]];
   U[[Uindex]]
   ];
a[m_]:=Module[{Aindex=m+1},
   If[Aindex>Alength,ExpandA[Aindex]];
   If[IntegerQ[x*(p*A[[Aindex-1]]-q*A[[Aindex-2]])]&&x*(p*A[[Aindex-1]]-q*A[[Aindex-
2]])≠0,Flag=Flag<>"B";,Flag=Flag<>"A";];
   A[[Aindex]]
   ];
```

```
isPeriodic[s_]:=Module[{positions,first=First[s],second=s[[2]],seconds,
out={0,0},periods},
   positions=Position[s,first]//Flatten;
   positions=Select[positions,2<#<Length[s]&];
   periods=Select[positions+1,s[[#]]□second&];
   If[Length[periods]>0,
    out={1,First[periods]-2};
    ];
   out
   ];
```

A.1 Mathematica Code for Products with One $\boldsymbol{B}$

This section contains the code used to achieve the results outlined in Section 4.1. It was run in conjunction with PeriodicFunctions.nb.

```
X[m_]:=(1+q*u[m])/(u[m+2]-q^(m+1));
periodic=0;
nonperiodic=0;
undefined=0;
disregard=0;
 For[p=1,p≤20,p++,
  For[q=-20,q≤20,q++,
    Ureset[];
    If[q□0,Continue[]];
    For[n=1,n≤30,n++,
     If[(u[n+2]-q^(n+1))□0,
      undefined++;
      Print["p=",p,", q=",q,", n=",n];
      Continue[];
      ];
     x=X[n];
     g=GCD[q*u[n]+1,u[n+1]];
     If[g□0|| x□1||x□0||x□-1,
      disregard++;
      Continue[]];
     Areset[(u[n+1])/g,(q*u[n]+1)/g];
     Flag="";
     output=Table[a[i],{i,0,n+2}];
     If[isPeriodic[output][[1]]□1,periodic++,nonperiodic++]
     ];
    ];];
periodic
nonperiodic
undefined
disregard
```

A.2 Mathematica Code for Products with Two $\boldsymbol{B}$'s

This section contains the code used to achieve the results outlined in Section 4.2. It was run in conjunction with PeriodicFunctions.nb.

```
f[x_]:=(x^2)*(u[m+2]*u[n+2]-(q^(n+m+2)))+x*(-2q*u[m+1]*u[n+1])+(q^2)*u[m]*u[n]-1;
periodic=0;
nonperiodic=0;
irrational=0;
rational=0;
disregard=0;
Periodic=List[];
Nonperiodic=List[];
Irrational=List[];
Disregard=List[];
 For[p=1,p≤20,p++,
  For[q=-20,q≤20,q++,
    Ureset[];
    If[q□0,Continue[]];
    For[m=0,m≤30,m++,
     For[n=m,n≤30,n++,
       If[Length[Solve[f[t]□0,t,Rationals]]>0,
        rational++;
        s=t/.Solve[f[t]□0,t,Rationals];,
        AppendTo[Irrational,{p,q,m,n}];
        irrational++;
        Continue[];
        ];
       For[i=1,i≤Length[s],i++,
        x=s[[i]];
        g=GCD[(q*x*u[m+1]*u[n+1]-q*q*u[n]u[m]+1),(x*u[m+1]*u[n+2]-q*u[n+1]u[m])];
        If[g□0||!NumberQ[x],Continue[]];
        If[x□1||x□0||x==-1,
         AppendTo[Disregard,{p,q,m,n}];
         disregard++;
         Continue[];
         ];
        Areset[(x*u[m+1]*u[n+2]-q*u[n+1]u[m])/g,
         (q*x*u[m+1]*u[n+1]-q*q*u[n]u[m]+1)/g];
        Flag="";
        output=Table[a[i],{i,0,n+m+3}];
        If[isPeriodic[output][[1]]□1,
         AppendTo[Periodic,{p,q,m,n,x,output[[1;;isPeriodic[output][[2]]+2]]}];
         periodic++,
         AppendTo[Nonperiodic,{p,q,m,n,x,output[[1;;isPeriodic[output][[2]]+2]]}];
         nonperiodic++];
        ];
       ];
     ];
    ];
  ];
 periodic
nonperiodic
irrational
rational
disregard
Export["C:\\Users\\mathoffice\\Desktop\\Final Runs\\nonperiodic2Bs.xls",Nonperiodic];
Export["C:\\Users\\mathoffice\\Desktop\\Final Runs\\periodic2Bs.xls",Periodic];
```

```
Export["C:\\Users\\mathoffice\\Desktop\\Final Runs\\irrational2Bs.xls",Irrational];
```

A.3 Mathematica Code for Products with Three $\boldsymbol{B}$'s

This section contains the code used to achieve the results outlined in Section 4.3. Due to the higher complexity, the periodic solutions were calculated separately from determining the number of rational roots. Factor3.nb was used to determine how many rational roots occurred in each case. Periodic3.nb used the rational x values to find periodic solutions. Both programs were run in conjunction with PeriodicFunctions.nb.

Factor3.nb

```
f[x_]:=(x^3)*(u[k+2]*u[m+2]*u[n+2]-(q^(k+n+m+3)))-
q*(x^2)*(u[k+1]*u[m+1]*u[n+2]+u[k+1]*u[m+2]*u[n+1]+u[k+2]*u[m+1]*u[n+1])+(q^2)*x*(u[k+
1]*u[m+1]*u[n]+u[k+1]*u[m]*u[n+1]+u[k]*u[m+1]*u[n+1])-(q^3)*u[k]*u[m]*u[n]-1;
nofactor=0;
onefactor=0;
twofactor=0;
threefactor=0;
solutions=0;
total=0;
 Monitor[For[p=1,p≤20,p++,
   For[q=-20,q≤20,q++,
     Ureset[];
     If[q==0||(p□q□1),Continue[]];
     For[k=0,k≤30,k++,
      For[m=k,m≤30,m++,
        For[n=m,n≤30,n++,
          total++;
          solution=Solve[f[t]□0,t];
          firstN=t/.solution[[1,1]];
          secondN=t/.solution[[2,1]];
          If[Length[solution]□3,thirdN=t/.solution[[3,1]],thirdN="na"];
          solutions=0;

If[NumberQ[firstN]&&Element[firstN,Rationals],solutions++];If[NumberQ[secondN]&&Elemen
t[secondN,Rationals],solutions++];If[NumberQ[thirdN]&&Element[thirdN,Rationals],soluti
ons++];
          Switch[solutions,0,nofactor++,1,onefactor++,2,twofactor++,3,threefactor++];
          ];
        ];
      ];
     ];
   ];,{p,q,k,m,n}]
nofactor
onefactor
twofactor
threefactor
total
```

Periodic3.nb

```
f[x_]:=(x^3)*(u[k+2]*u[m+2]*u[n+2]-(q^(k+n+m+3)))-
```

```
q*(x^2)*(u[k+1]*u[m+1]*u[n+2]+u[k+1]*u[m+2]*u[n+1]+u[k+2]*u[m+1]*u[n+1])+(q^2)*x*(u[k+
1]*u[m+1]*u[n]+u[k+1]*u[m]*u[n+1]+u[k]*u[m+1]*u[n+1])-(q^3)*u[k]*u[m]*u[n]-1;
data=List[];
 For[p=1,p≤20,p++,
  AppendTo[data,{}];
  AppendTo[data[[p]],{"n","m","k","p","q","period","","An","X"}];
  For[q=-20,q≤20,q++,
   Ureset[];
   If[q□p□1||q==0,Continue[]];
   For[k=0,k≤30,k++,
    For[m=0,m≤30,m++,
      For[n=0,n≤30,n++,

If[Length[Solve[f[t]□0,t,Rationals]]>0,s=t/.Solve[f[t]□0,t,Rationals],Continue[]];
        For[i=1,i≤Length[s],i++,
         x=s[[i]];
         g=GCD[((x*u[k+2]*(x*u[n+1]*u[m+2]-q*u[n]u[m+1]))+u[k+1]*((-
q*x*u[m+1]*u[n+1]+q*q*u[n]u[m]))),
           ((q*x*u[k+1]*(x*u[n+1]*u[m+2]-q*u[n]u[m+1]))+q*u[k]*((-
q*x*u[m+1]*u[n+1]+q*q*u[n]u[m]))+1)];
         If[g□0||!NumberQ[x]|| x==-2||x□1||x□0||x==-1,Continue[]];
         Areset[((x*u[k+2]*(x*u[n+1]*u[m+2]-q*u[n]u[m+1]))+u[k+1]*((-
q*x*u[m+1]*u[n+1]+q*q*u[n]u[m])))/g,
          ((q*x*u[k+1]*(x*u[n+1]*u[m+2]-q*u[n]u[m+1]))+q*u[k]*((-
q*x*u[m+1]*u[n+1]+q*q*u[n]u[m]))+1)/g];
         Flag="";
         output=Table[a[i],{i,0,(k+m+n+4)}];

If[isPeriodic[output][[1]]□1,AppendTo[data[[p]],{n,m,k,p,q,isPeriodic[output],Flag,out
put[[1;;isPeriodic[output][[2]]+2]],StringForm[x]}]
          ];
         ];
        ];
      ];
    ];]
  ];
Export["C:\\Users\\mathoffice\\Desktop\\Final Runs\\threeBsmore.xls",data];
```

A.4 Mathematica Code for Products with Four $\boldsymbol{B}$'s

This section contains the code used to achieve the results outlined in Section 4.4. Once again, the periodic solutions were calculated separately from determining the number of rational roots. Factor4.nb was used to determine how many rational roots occurred in each case. Periodic4.nb used the rational x values to find periodic solutions. Both programs contain the necessary code from PeriodicFunctions.nb, thus each program was run without an initialization program.

Factor4.nb

```
Usize=200;
Ureset[]:=Module[{},
   U=Table[0,{i,1,Usize}];
   U[[1]]=0;U[[2]]=1;
   Ulength=2;
```

```
   ];
ExpandU[index_]:=Module[{i},
   For[i=Ulength+1,i≤ index,i++,
     Ulength++;
     U[[Ulength]]=p*U[[Ulength-1]]-q*U[[Ulength-2]];
     ];];
u[m_]:=Module[{Uindex=m+1},
   If[Uindex>Ulength,ExpandU[Uindex]];
   U[[Uindex]]];
f[x_]:=(x^4)*a-q*(x^3)*b+(q^2)*(x^2)*c-(q^3)*x*d+e;
none=0;
one=0;
two=0;
three=0;
four=0;
total=0;
 Monitor[For[p=1,p≤20,p++,
    For[q=-20,q≤20,q++,
      Ureset[];
      If[q□0||(p□q□1),Continue[];];
      For[j=0,j≤20,j++,
       For[k=j,k≤20,k++,
         For[m=j,m≤20,m++,
           For[n=j,n≤20,n++,
             total++;
             a=(u[j+2]*u[k+2]*u[m+2]*u[n+2]-(q^(j+k+n+m+4)));

b=(u[j+2]*u[k+1]*u[m+1]*u[n+2]+u[j+1]*u[k+2]*u[m+2]*u[n+1]+u[j+2]*u[k+2]*u[m+1]*u[n+1]
+u[j+1]*u[k+1]*u[m+2]*u[n+2]);

c=(u[j+1]*u[k]*u[m+1]*u[n+2]+u[j]*u[k+1]*u[m+2]*u[n+1]+u[j+1]*u[k+2]*u[m+1]*u[n]+u[j+2
]*u[k+1]*u[m]*u[n+1]+2*u[j+1]*u[k+1]*u[m+1]*u[n+1]);

d=(u[j+1]*u[k]*u[m]*u[n+1]+u[j+1]*u[k+1]*u[m]*u[n]+u[j]*u[k+1]*u[m+1]*u[n]+u[j]*u[k]*u
[m+1]*u[n+1]);
             e=(q^4)*u[j]*u[k]*u[m]*u[n]-1;
             fact=FactorList[f[x]];
             If[a≠0,
              If[Length[fact]□5,
               four++;
               Continue[];];
              If[Length[fact]□4&&Exponent[fact[[4,1]],x]≠2,
               three++;
               Continue[];];

If[(Length[fact]□2&&fact[[2,2]]□4)||(Length[fact]□3&&Exponent[fact[[3,1]],x]□3)||(Leng
th[fact]□3&&fact[[2,2]]□2&&Exponent[fact[[3,1]],x]□2),
               one++;
               Continue[];];

If[(Length[fact]□4&&Exponent[fact[[4,1]],x]□2)||(Length[fact]□3&&fact[[2,2]]□1&&fact[[
3,2]]□3)||(Length[fact]□3&&fact[[2,2]]□3&&fact[[3,2]]□1)||(Length[fact]□3&&fact[[2,2]]
□2&&fact[[3,2]]□2),two++;Continue[];];

If[(Length[fact]□2&&fact[[2,2]]□2&&Exponent[fact[[2,1]],x]□2)||(Length[fact]□2&&Expone
nt[fact[[2,1]],x]□4)||(Length[fact]□3&&Exponent[fact[[2,1]],x]□2&&Exponent[fact[[3,1]]
,x]□2),
               none++;
               Continue[];];
              ,
```

```
              If[b≠0,
                If[Length[fact]□4,three++;Continue[];];
                If[Length[fact]□2&&fact[[2,2]]□1,none++;Continue[];];

If[(Length[fact]□2&&fact[[2,2]]□3)||(Length[fact]□3&&Exponent[fact[[3,1]],x]□2),one++;
Continue[];];
                If[Length[fact]□3&&Exponent[fact[[3,1]],x]≠2,two++;Continue[];];
                ,
                If[c≠0,
                  If[Length[fact]□3,two++;Continue[];];
                  If[Length[fact]□2&&fact[[2,2]]□2,one++;Continue[];];
                  If[Length[fact]□2&&Exponent[fact[[2,1]],x]□2,none++;Continue[];];
                  ,
                  If[d≠0,one++;Continue[];,none++;Continue[];];];];];
             Print[fact,"p=",p,"q=",q,"n=",n,"m=",m,"k=",k,"j=",j];
             ];];];];];];;,{p,q,n,m,k,j}];
none
one
two
three
four
total
```

Periodic4.nb

```
CloseKernels[];
 machineOpt={
   {"blitzen",11},
   {"rigel",5},
   {"sirius",11},
   {"comet",10},
   {"merak",11},
   {"santa",12},
   {"zeus",23}
   };
 Needs["SubKernels`RemoteKernels`"]
 LaunchKernels[RemoteMachine[#[[1]],#[[2]]]]&/@machineOpt;
 saveTask=CreateScheduledTask[FrontEndExecute[FrontEndToken["Save"]],5*60];
StartScheduledTask[saveTask];
ParallelEvaluate[
  $MaxExtraPrecision=1000;
  $HistoryLength=0;
  Asize=200;
  Areset[a1_,a2_]:=Module[{},
    A=Table[0,{i,1,Asize}];
    A[[1]]=a1;A[[2]]=a2;
    Alength=2;
    ];
  ExpandA[index_]:=Module[{i},
    For[i=Alength+1,i≤ index,i++,
      Alength++;
      A[[Alength]]=Piecewise[{{x*(p*A[[Alength-1]]-q*A[[Alength-
2]]),IntegerQ[x*(p*A[[Alength-1]]-q*A[[Alength-2]])]}},p*A[[Alength-1]]-q*A[[Alength-
2]]]];
    ];
  a[m_]:=Module[{Aindex=m+1},
    If[Aindex>Alength,ExpandA[Aindex]];
    A[[Aindex]]
    ];
  isPeriodic[s_]:=Module[{positions,first=First[s],second=s[[2]],seconds,
out={0,0},periods},
    positions=Position[s,first]//Flatten;
```

```
    positions=Select[positions,2<#<Length[s]&];
    periods=Select[positions+1,s[[#]]□second&];
    If[Length[periods]>0,
     out={1,First[periods]-2};
     ];
    out
    ];

  Amat={{p,-q},{1,0}};
  B={{p*x,-q*x},{1,0}};

W=B.MatrixPower[Amat,n].B.MatrixPower[Amat,m].B.MatrixPower[Amat,k].B.MatrixPower[Amat
,j];

  disregard=0;
  periodic=0;
  nonperiodic=0;
  counter=0;
  iDontCare=0;

  DrittIMidten[vector_]:=
   Block[{system,xvaluesone,sol,xvaluestwo,i,solution,gcd,output},
    Check[
      p=vector[[1]];
      q=vector[[2]];
      j=vector[[3]];
      k=vector[[4]];
      m=vector[[5]];
      n=vector[[6]];
      If[q□0||p□q□1||(n□m□j□k),Return[]];
      counter++;
      Clear[x];
      Quiet[Check[system=Eigensystem[W],Return[]]];
      system[[2]]=Select[system[[2]],#≠  {0,0}&]; (* select nonzero eigenvectors *)
      iDontCare+=Length[Select[system[[2]],#□{0,0}&]];
      (*Print[system];*)

      If[Length[system[[2]]]□0,Return[]]; (* Nothing to work with *)
      system[[2]]=Simplify[#/PolynomialGCD[#[[1]],#[[2]]]&/@system[[2]]];

xvaluesone=Select[Solve[system[[1,1]]□1,x],Element[#[[1,2]],Rationals]&&NumberQ[#[[1,2
]]]&];
      sol=Partition[Riffle[xvaluesone, system[[2,1]]/.xvaluesone],2];
      If[Length[system[[2]]]>1,
       xvaluestwo=Select[Solve[system[[1,2]]□1,x],
         Element[#[[1,2]],Rationals]&&NumberQ[#[[1,2]]]&];
       sol=Union[sol,Partition[Riffle[xvaluestwo, system[[2,2]]/.xvaluestwo],2]];
       ];
      If[Length[sol]≥1,
       If[sol[[1]]□{},
        sol=Drop[sol,1];
        ]
       ];
      For[i=1,i≤Length[sol],i++,
       solution=sol[[i]];
       x=x/.solution[[1]];
       If[x□1||x□-1||x□0,
        disregard++;
        Continue[];];
       If[solution[[2]]□{0,0},
        iDontCare++;
```

```
        Continue[];];
       gcd=GCD[solution[[2,1]],solution[[2,2]]];
       solution[[2]]/=gcd;
       Areset[solution[[2,2]],solution[[2,1]]];
       output=Table[a[i],{i,0,n+m+k+j+4}];
       If[isPeriodic[output][[1]]□1,
        periodic++;
        Print["p=",p," q=",q," n=",n," m=",m," k=",k," j=",j," x=",x,output[[1;;2]]];,
        nonperiodic++];
       ];
      ,Print["ERROR: p=",p," q=",q," n=",n," m=",m," k=",k," j=",j]];
    ];
  ];
 T=Flatten[Table[{p,q,j,k,m,n},{p,1,1},{q,-
20,20},{j,0,20},{k,j,20},{m,j,20},{n,j,20}],5];
len=Length[T];
positionNext=100000;
Print["Length = ",len];
Monitor[
 For[i=0,i≤ len/100000,i++,
   positionNext=(i+1)*100000;
   If[positionNext≥ len,positionNext=len];
   T2=T[[i*100000+1;;positionNext]];
   SetSharedVariable[T2];
   Parallelize[
     DrittIMidten/@T2;
     ]
    ParallelEvaluate[ClearSystemCache[]];
   ];,Refresh[i*100000,UpdateInterval→1]]
 counter=Total[ParallelEvaluate[counter]];
disregard=Total[ParallelEvaluate[disregard]];
nonperiodic=Total[ParallelEvaluate[nonperiodic]];
periodic=Total[ParallelEvaluate[periodic]];
iDontCare=Total[ParallelEvaluate[iDontCare]];
periodic
nonperiodic
disregard
iDontCare
counter
CloseKernels[];
```