University of Minnesota

Patterns of Non-Simple Continued Fractions

Pablo Mello

advised by
Dr. John Greene

August 26, 2017

Contents

1 Introduction

1.1 Simple Continued Fractions

Suppose we are trying to solve the following quadratic equation:

$$x^2 - 9x - 1 = 0.$$

Dividing by x we can rewrite it as

$$x = 9 + \frac{1}{x}.$$

Notice that the variable x is still found on the right-hand side of this equation and thus can be replaced by its equal, namely $9 + \frac{1}{x}$, which gives

$$x = 9 + \frac{1}{9 + \frac{1}{x}}.$$

We can continue this procedure indefinitely to produce a never-ending staircase of fractions:

$$x = 9 + \frac{1}{9 + \frac{1}{9 + \frac{1}{9 + \dots}}}. \tag{1.1.1}$$

At first glance, as x continues to appear on the right-hand side of this continued fraction, it does not seem to be getting any closer to the solution of the equation. It turns out, however, it contains a succession of fractions,

$$9, \quad 9 + \frac{1}{9}, \quad 9 + \frac{1}{9 + \frac{1}{9}}, \quad 9 + \frac{1}{9 + \frac{1}{9 + \frac{1}{9}}}, \quad \dots \quad ,$$

which are obtained by stopping at consecutive stages. These numbers, which are called convergents [3, p. 6], when converted into fractions give better and better approximations to the positive root of the given quadratic equation

$$9, \quad \frac{82}{9} = 9.111..., \quad \frac{747}{82} = 9.10975..., \quad \frac{6805}{474} = 9.10977....$$

The quadratic formula gives the actual root

$$x = \frac{9 + \sqrt{85}}{2} = 9.10977...$$

which is consistent with the last result above. Multiple-decked fractions like (1.1.1) are called continued fractions. Writing a number using such a representation seems very insignificant. It turns out, however, this kind of representation provides much insight into many mathematical problems, particularly into the nature of numbers [3, p. 3]. We must start with the basics first, however. We start by introducing some definitions. An expression of the form

$$x = a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}} \tag{1.1.2}$$

is called a continued fraction. In general, the numbers $a_0, a_1, a_2, ..., b_1, b_2, b_3, ...$ may be any real or complex numbers, and the number of terms may be finite or infinite. Furthermore, if a continued fraction has $b_1, b_2, b_3, ...$ all equal to 1, then it is called a simple continued fraction. An infinite simple continued fraction has, therefore, the following form

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}, \tag{1.1.3}$$

where a_0 is an integer and the terms $a_1, a_2, a_3, ...$ are positive integers. It is convenient to denote (1.1.3) by the compact symbol $[a_0, a_1, a_2, ...]$, where the terms $a_0, a_1, a_2, ...$ are called the partial quotients of the continued fraction [3, p. 6]. A finite simple continued fraction has only a finite number of terms $a_0, a_1, ..., a_{n-1}$, and it can be compactly written as $[a_0, a_1, ..., a_{n-1}]$. Such a fraction is also called a terminating continued fraction. It can be proven that every rational number can be expressed as a finite simple continued fraction [3, p. 14, Theorem 1.1].

Example 1.1.1 The following is an example of a finite continued fraction

$$\frac{67}{29} = 2 + \cfrac{1}{3 + \frac{1}{4 + \frac{1}{2}}} = [2, 3, 4, 2].$$

The above result can be obtained through the following algorithm: First we divide 67 by 29 to get the quotient 2 and the remainder 9, so that

$$\frac{67}{29} = 2 + \frac{9}{29} = 2 + \frac{1}{\frac{29}{9}}.$$

Next, we replace $\frac{9}{29}$ by the reciprocal of $\frac{29}{9}$ so we get the numerator 1. Then, we divide 29 by 9 to obtain

$$\frac{29}{9} = 3 + \frac{2}{9} = 3 + \frac{1}{\frac{9}{2}}.$$

Finally, we divite 9 by 2 to obtain

$$\frac{9}{2} = 4 + \frac{1}{2}$$

at which point the process ends. Putting the equations above together we get

$$\frac{67}{29} = 2 + \frac{1}{\frac{29}{9}} = 2 + \cfrac{1}{3 + \frac{1}{\frac{9}{2}}} = 2 + \cfrac{1}{3 + \frac{1}{4 + \frac{1}{2}}},$$

or

$$\frac{67}{29} = [2, 3, 4, 2].$$

1.2 Non-simple Continued Fractions

Simple continued fractions were studied at great length by mathematicians of the seventeenth and eighteenth centuries and are a subject of active investigation today. However, not much attention has been given to non-simple continued fractions. The difference between simple and non-simple continued fractions is that at least one of numerators of a non-simple continued fraction must be a real number different from 1. Non-simple continued fractions can, therefore, be generally expressed in the following manner:

$$x = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \frac{b_4}{\cdots}}}}, \text{for at least one } b_i \neq 1. \tag{1.2.1}$$

In this monograph, however, we shall restrict our discussion to the case when all b's are equal to a real number z. We, therefore, investigate continued fractions of the form

$$x = a_0 + \frac{z}{a_1 + \frac{z}{a_2 + \frac{z}{a_3 + \frac{z}{\cdots}}}}, z \neq 1, \tag{1.2.2}$$

where a_0 is an interger, a_j, for $j > 0$, are positive integers and z is a real number. We shall use a more compact form to write (1.2.2):

$$x = [a_0, a_1, a_2, a_3, ...]_z.$$

A paper by Maxwell Anselm and Steven Weintraub [1] investigated this generalization when z is replaced by an arbitrary positive integer other than 1. Another paper by Jesse Schmieg [4] generalized further to the case of an arbitrary real number $z \geq 1$. He mainly focused on the case where z is rational but not an integer, especially when x, the number to be expanded, is also rational.

In order to generate continued fractions or expand a number into continued fractions, we use the continued fraction algorithm. This algorithm takes two parameters, namely a fixed real number z and a real number x, which is the number to be expanded. The output is a list $[a_0, a_1, a_2...]$ which satisfies the relation in (1.2.2). There is a few ways of finding numbers $a_0, a_1, a_2...$, but here we use the following approach: we let $x = x_0$ and recursively do the expansion as follows:

$$x = a_0 + \frac{z}{x_1} = a_0 + \frac{z}{a_1 + \frac{z}{x_2}} = ...$$

We let $a_0 = \lfloor x_0 \rfloor$ and solve for x_1: $x_1 = \frac{z}{x_0 - a_0}$. We then iterate this process: $a_1 = \lfloor x_1 \rfloor$, $x_2 = \frac{z}{x_1 - a_1}$, and so on. If it were to happen that x_n is an integer for some n, then $a_n = x_n$, and the algorithm would stop. Otherwise it continues indefintely.

Example 1.2.1 Expansion of x = $\frac{3}{2}$ with z = $\sqrt{2}$

$$x_0 = \frac{3}{2}, a_0 = 1, \quad \rightarrow \quad x_1 = \frac{\sqrt{2}}{\frac{3}{2} - 1} = 2\sqrt{2}, a_1 = 2, \quad \rightarrow$$

$$x_2 = \frac{\sqrt{2}}{s\sqrt{2} - 2} = \frac{2 + \sqrt{2}}{2}, a_2 = 1 \quad \rightarrow \quad x_3 = \frac{\sqrt{2}}{\frac{2+\sqrt{2}}{2} - 1} = \frac{2\sqrt{2}}{\sqrt{2}} = 2, a_3 = 2.$$

This expansion is finite, and the algorithm outputs $a = [a_0, a_1, a_2, a_3] = [1, 2, 1, 2]$. Alternatively, we can have both the number being expanded and z be fractional numbers.

Example 1.2.2 Expansion of $\frac{20}{7}$ with $z = \frac{3}{2}$.

$$\frac{20}{7} = 2 + \frac{6}{7} = 2 + \frac{3/2}{\frac{7}{4}} = 2 + \frac{3/2}{1 + \frac{3/2}{2}},$$

so $\frac{20}{7} = [2, 1, 2]_{3/2}$, which is a finite expansion.

The expansions above are both finite, but one of the most interesting characteristics of continued fractions arguably is the fact that their expansions can be periodic.

Example 1.2.3 We derive the expansion of $x = \sqrt{2}$ and $z = \frac{3}{2}$.

$$a_0 = 1, x_1 = \frac{3/2}{x_0 - a_0} = \frac{3/2}{\sqrt{2} - 1} = \frac{3/2(\sqrt{2} + 1)}{2 - 1} = \frac{3\sqrt{2} + 3}{2}.$$

Next, $a_1 = \lfloor x_1 \rfloor = 3$ and

$$x_2 = \frac{3/2}{\frac{3\sqrt{2}+3}{2} - 3} = \frac{3}{3\sqrt{2} - 3} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1.$$

Then, $a_2 = \lfloor x_2 \rfloor = 2$ and $x_3 = \frac{3/2}{\sqrt{2}-1} = \frac{3\sqrt{2}+3}{2} = x_1$. This means that we get periodic behavior: the x's follow the pattern $[\sqrt{2}, \frac{3\sqrt{2}+3}{2}, \sqrt{2} + 1, \frac{3\sqrt{2}+3}{2}, \sqrt{2} + 1, ...]$, and the a's follow the pattern $[1, 3, 2, 3, 2, 3, ...]$. We can also write it as $\sqrt{2} = [1, \overline{3, 2}]_{\frac{3}{2}}$ and say that the expansion is periodic with period length 2.

Furthermore, the special case of purely periodic behavior with period 1 is especially nice. If we, for example, start off with an equation of the form $x^2 - bx - z = 0$ then $x = b + \frac{z}{x}$, which means x has a purely periodic z-expansion with period length 1.

Example 1.2.4 We find approximations to the positive solution to $x^2 + 3x - 7 = 0$. First, we need to make this quadratic equation take the form $x^2 - bx - c = 0$ for $|b| >= |c|$. If we let $x = y - 4$, then we get $y^2 - 5y - 3 = 0$, so $y = 5 + \frac{3}{y}$ leads to an expasion for y with z = 3.

$$y = 5 + \frac{3}{y} = 5 + \frac{3}{5 + \frac{3}{y}} = 5 + \frac{3}{5 + \frac{3}{5 + \frac{3}{\cdots}}} \quad \text{or} \quad y = [5, 5, 5, ...]_3.$$

If we subtract 4, we get

$$x = [1, 5, 5, ...]_3 \quad \text{or} \quad x = [1, \overline{5}]_3.$$

The above leads to approximate solutions

$$1, \quad 1 + \frac{3}{5} = 1.6, \quad 1 + \frac{3}{5 + \frac{3}{5}} = 1.53571..., \quad 1 + \frac{3}{5 + \frac{3}{5+\frac{3}{5}}} = 1.54194..., \quad \ldots$$

which approach the exact solution 1.54138... .

As discussed above, continued fractions can have three kinds of expansions: finite expansions, periodic expansions or infinite non-periodic expansions. Different values of x and z will give different kinds of expansions. However, from this point on, we shall mainly focus on the cases when the number to be expanded is a fraction and z is a radical, investigating periodic behaviors with various period lengths. The kind of non-simple continued fractions that we shall discuss can, therefore, be generally expressed in the following way:

$$x = a_0 + \frac{\sqrt{n}}{a_1 + \frac{\sqrt{n}}{a_2 + \frac{\sqrt{n}}{a_3 + \frac{\sqrt{n}}{\cdots}}}}.$$

As we shall demonstrate in subsequent sections, non-simple continued fractions with the above characteristics can produce the three kinds of expansions; however, we give especial attention to periodic expansions.

2 Periodic Results

2.1 General Results

We start by formally defining a recursive algorithm for computing continued fractions with real numerators and verifying that the algorithm gives successful results.

Definition. Given $x \in \mathbb{R}$ and $z \in \mathbb{R}$ with $z \geq 1$, choose sequences (x_j) and (a_j) by

$$x_0 = x, \qquad a_j = \lfloor x_j \rfloor, \qquad x_{j+1} = \frac{z}{x_j - a_j},$$

for $j = 0, 1, 2, ...$, and terminating when x_j is an interger. This creates the expansion

$$x = [a_0, a_1, a_2, ..., a_j]_z, \tag{2.1.1}$$

if the expansion terminates. Otherwise, we write

$$x = [a_0, a_1, a_2, ...]_z. \tag{2.1.2}$$

The above is called the maximal expansion of x with respect to z.

Theorem 2.1.1. *If x is a noninteger real number, then it has an expansion $x = [a_0, a_1, ..., a_k, x_{k+1}]_z$, for all $k \geq 0$ for which the expression is valid.*

Proof. We proceed by induction. It is clear that $x = [x] = [x_0]$. We assume x has an expansion $[a_0, ..., a_{k-1}, x_k]_z$. If x_k is not an integer, then $x_{k+1} = \frac{z}{x_k - a_k}$. This can be rewritten as $x_k = a_k + \frac{z}{x_{k+1}} = [a_k, x_{k+1}]$, where $a_k = \lfloor x_k \rfloor$. Therefore, $x = [a_0, ..., a_{k-1}, x_k]_z = [a_0, a_1, ..., a_k, x_{k+1}]_z$. ■

Theorem 2.1.2. *If x has maximal expansion $[a_0, a_1, ...]$, then for all k = 0, 1, 2, ..., $x_k = [a_k, a_{k+1}, ...]$.*

Proof. We let $y = x_k$ and, if x_k is not an integer, then y has an expansion $[b_0, b_1, b_2, ...]$. Notice that $y = y_0 = x_k, y_1 = x_{k+1}, ...$. Therefore, $b_0 = \lfloor y_0 \rfloor = \lfloor x_k \rfloor = a_k$, $b_1 = \lfloor y_1 \rfloor = \lfloor x_{k+1} \rfloor = a_{k+1}$, and so on. ■

Theorem 2.1.3. *If x has an expansion $x = [a_0, a_1, ..., a_{k-1}, ...]_z$, then $a_k \geq \lfloor z \rfloor$ for all $k > 0$, and if the expansion is finite, $x = [a_0, a_1, ..., a_k]_z$, then the final partial quotient satisfies $a_k > z$.*

Proof. We assume that $k > 0$, so we have $a_{k-1} = \lfloor x_{k-1} \rfloor$, which means $0 < x_{k-1} - a_{k-1} < 1$. Consequently, $x_k = \frac{z}{x_{k-1} - a_{k-1}} > z$. Since $a_k = \lfloor x_k \rfloor$, and $x_k > z$, it must be that $a_k \geq \lfloor z \rfloor$. If x has a finite expansion, then x_k must be an integer, which means $a_k = \lfloor x_k \rfloor = x_k > z$. ■

2.2 Periodicity

As mentioned earlier, some continued fraction expansions eventually become periodic. In this section, we go into further detail about periodic bahaviors and find interesting results. Periodic continued fraction expansions consist of a leading tail and a period, both of which can vary in length. If an expansion is periodic with period n and tail length k, then it has the form

$$[a_0, a_1, ..., a_{k-1}, \overline{a_k, a_{k+1}, ..., a_{k+n-1}}]_z \tag{2.2.1}$$

with the part under the line repeating indefinitely. Periodicity can be proven, as demonstrated next.

Example 2.1.1 We use the continued fraction algorithm, as described in section 2.1, to compute the continued fraction expansion of $\frac{3}{2}$ with $z = \sqrt{8}$. We first set $x_0 = \frac{3}{2}$, and then calculate the floor function a_0 of x_0, so we obtain

$$x_0 = a_0 + \frac{z}{x_1} = \frac{3}{2} = 1 + \frac{\sqrt{8}}{x_1}.$$

We then recursively continue expanding

$$x_1 = 4\sqrt{2}, a_1 = 5 \quad \rightarrow \quad x_2 = \frac{2}{7}(8 + 5\sqrt{2}), a_2 = 4 \quad \rightarrow \quad x_3 = 5 + 3\sqrt{2}, a_3 = 9 \quad \rightarrow$$

$$x_4 = 6 + 4\sqrt{2}, a_4 = 11 \quad \rightarrow \quad x_5 = \frac{2}{7}(8 + 5\sqrt{2}), a_5 = 4.$$

At this point, we are able to prove periodicity. Notice that x_2 and x_5 are equal, which means the term $\frac{2}{7}(8 + 5\sqrt{2})$ reappears in the (x_n) sequence, and therefore, the steps will repeat from that point forward, causing period length 3. Indeed, $\frac{3}{2} = [1, 5, \overline{4, 9, 11}]_{\sqrt{8}}$. In general, we can say that x will have periodic behavior if and only if $x_i = x_j$ for some $i < j$. See Theorem 3.1.1 for a proof.

Remark. In this research, we use Floyd's Cycle Finding Algorithm to find periods and compute their lengths, as described in the appendix.

2.3 Search for Periodic Behavior

Using the algorithm defined in setion 2.1, we searched for periodic behaviors in expansions of various rational values with different values of n, for $z = \sqrt{n}$. We used the following approach to define these values: x, which is the value to be expanded, is a fraction x = $\frac{u}{v}$ such that u and v are integers, where v ranges from 2 to 500, and u ranges from $v + 1$ to $2v - 1$ and is relatively prime to v. The reason for the constraint on u is that if $y = x + k$, where k is a positive integer, and if $x = [a_0, a_1, a_2, a_3, ...]$, then $y = [a_0 + k, a_1, a_2, a_3, ...]$. That is to say, if two numbers differ by an integer, their expansions are identical except for a_0. Consequently, they will have the same period, if they are periodic, and both will have finite expansions, if either one of them does. This means it is unnecessary to look at, for instance, the expansion of 19/7 if we have already investigated the expansions of $12/7 = [1, 3, 2, 2, 2, 3, ...]_{\sqrt{8}}$ or $5/7 = [0, 3, 2, 2, 2, 3, ...]_{\sqrt{8}}$. Hence, we arbitrarily fix a_0 to be 1, which means we need $1 \leq \frac{u}{v} \leq 2$, and therefore $v + 1 \leq u \leq 2v - 1$. With this constraint applied, we expanded fractions $\frac{u}{v}$ for each value of z = $\sqrt{n}$ for $n < 100$ (excluding those that are perfect squares).

The two tables below summarize the result of our search. The first table is ordered by the length of the periods found. Its first column indicates the length of the periods, and its second column indicates the values of n (for z = $\sqrt{n}$) used when doing the expansions. The second table provides the same information but ordered by the values of n in which we found periodic behavior.

Table 2.3.1: Ordered by period length

length	**n**
0	2, 3, 5, 6, 7, 8, 12
1	2, 6, 12, 32, 60, 96
2	3, 8, 15, 24, 35, 48, 63, 80, 99
3	8
4	5
6	2, 8
8	3
12	3
13	12
15	6
22	10
72	3

Table 2.3.2: Ordered by n

n	length
2	0, 1, 6
3	0, 2, 8, 12, 72
5	0, 4
6	0, 1, 15
7	0
8	0, 2, 3, 6
10	22
12	0, 1, 13
15, 24	2
32	1
35, 48	2
60	1
63, 80	2
96	1
99	2

Remark. Period length 0 indicates that the expansion is terminating.

Some of the most stricking results we found were regarding the shape of the periods in the periodic expansions. Although we expanded a great number of rationals for various values of n, we only discuss in detail the cases of n where all rationals had either a finite or periodic expansion. We start by discussing expansions for $n = 2$. In this case, each period length presented only one kind of pattern. For instance, we expanded a total of 76115 rationals, 27% of which presented periodic behavior with period length 1, and all of them surprisingly became periodic with the number 3. That is to say, their expansions had the following form: $x = [a_0, a_1, ..., a_{k-1}, \overline{3}]_{\sqrt{2}}$, for some $k > 0$. The same was true for periodic expansions with period length 6, which only presented cyclic permutations of the period $[10, 19, 4, 3, 3, 17]$. The following is a table summarizing the frequency of each period length in expansions for $n = 2$:

Table 2.3.3: Frequency of period lengths for $n = 2$

period length	frequency
0	55056
1	21052
6	7

Next, we discuss expansions for $n = 3$. In this case, if we disregard permutations, periods also presented only one unique form. For instance, periodic expansions with period length 2 consisted of 42% of the expanded rationals, and all of them had cycles $[2, 4]$ or $[4, 2]$. Similarly, periodic expansions with period length 72 all had cyclic permutations of the following cycle: $[20, 65, 1, 1, 1, 1, 1, 2, 248, 5, 24, 2, 7, 19, 4, 8, 4, 3, 2, 14, 3, 49, 2, 9, 2, 3, 1, 2, 5, 5, 3, 8, 5, 2, 96, 7, 40, 20, 5, 3, 2, 5, 4, 2, 21, 2, 8, 4, 4, 12, 3, 9, 6, 2, 35, 10, 2, 4, 2, 3, 22, 1, 2, 6, 25, 2, 2, 3, 1, 2, 8, 2, 20]$. In fact, all permutations of periods we found were only cyclic permutations. The following is a table summarizing the frequency of each period length in expansions for $n = 3$:

Table 2.3.4: Frequency of period lengths for $n = 3$

period length	**frequency**
0	14287
2	31641
8	57
12	1394
72	28736

Expansions for $n = 6$ are remarkable as well. In this case, periodic expansions with period length 1 accounted for 72% of the expanded rationals, and all of them had cycle [5]. Furthermore, only two rationals, namely 98/81 and 283/196, presented the period $[3, 79, 2, 4, 15, 5, 3, 7, 2, 5, 6, 13, 6, 3, 8]$, which has length 15. The remaining rationals had finite expansions. The following is a table summarizing the frequency of each period length in expansions for $n = 6$:

Table 2.3.5: Frequency of period lengths for $n = 6$

period length	**frequency**
0	3514
1	8715
15	2

For most values of n, very few expanded rationals appeared to be periodic or finite within the limits of our search. However, a small number of these n's led to a significant number of periodic expansions. The following tables show lengths of periods that came up in expansions for each of these values of n, as well as their frequencies.

Table 2.3.6: Frequency of period lengths for $n = 8$

period length	**frequency**
0	36
2	495
3	99
6	1
other	2761

Table 2.3.7: Frequency of period lengths for $n = 12$

period length	**frequency**
0	99
1	219
13	37
other	11876

Here, "other" means that the search results were inconclusive. That is to say, we could not determine whether or not the expansion was terminating or periodic. The reason for this might lie upon the deepness of our search. We searched 500 steps deep for $n < 4$, 200 steps deep for $3 < n < 8$ and 100 steps deep for the rest. A deeper search for each n might have given different results.

For $n = 5$, almost all expansions appeared to be infinite and non-periodic. The exceptions being 344/321 and 521/356, both of which presented periodic expansions with length 4 and tails 11 and 24, respectively, and 26 other rationals, which had finite expansions. As we shall see in the next table, as n gets larger, we find less and less periodic behavior, and the distance between the n's that present period bahavior increases more and more.

We next give a table that displays the following three counts: terminating expansions, periodic expansions and "other" for each n on Table 2.3.2. Please, notice that we expanded rationals for all $n < 100$ that were not perfect squares, but only those listed on Table 2.3.8 had periodic behavior. For all the other values of n, for example when $n = 11$, there were no rationals that had either finite expansions or expansions that became periodic within the bounds of our search.

Table 2.3.8: Overall frequency for each n on Table 2.3.2

n	**terminating expansions**	**periodic expansions**	**other**	**total**
2	55056	21059	0	76115
3	14287	61828	0	76115
5	26	2	76087	76115
6	3514	8717	0	12231
7	1	0	12230	12231
8	36	595	11600	12231
10	0	6	12225	12231
12	99	256	11876	12231
15	0	1	3042	3043
24, 32	0	2	3041	3043
35, 48	0	1	3042	3043
60	0	2	3041	3043
63, 80	0	1	3042	3043
96	0	2	3041	3043
99	0	1	3042	3043

Up to this point, we have only discussed period lengths, but we would like to conclude this section by providing some data on finite expansions as well as tail lengths. The following are two tables: the first gives the number of rationals that led to finite expansions with each length ≤ 10, and the second gives the number of rationals that led to periodic expansions with each tail length ≤ 10, both of which limited to the bounds and deepness of our search:

Table 2.3.9: Number of expansions **x** finite expansion lengths

length	4	5	6	7	8	9	10
number of expansions	2	6	8	15	31	40	49

Table 2.3.10: Number of expansions **x** tail lengths

tail length	2	3	4	5	6	7	8	9	10
number of expansions	5	5	1	10	21	26	29	46	86

3 Purely Periodic Expansions

3.1 Expansion into Quadratic Surds

In the previous chapter, we mentioned that, when n=2, all periodic expansions with period length 1 had the form $x = [a_0, a_1, ..., a_{j-1}, 3, 3, 3, ...]$ for some interger $j > 0$. Since x has an expansion as in formula (2.2.1), we can say x_j is purely periodic, based on Theorem 2.1.2. This led us to the investigation of real numbers with purely periodic expansions.

Example 3.1.1 We show the values that x assume through the expansions $x = [3, 3, 3, ...]$ and $x = [4, 4, 4, ...]$ in the case with $n = 2$.

By Theorem 2.1.1, we have $x = [3, x]$ or $x = 3 + \frac{\sqrt{2}}{x}$. This leads to $x = 2 + \sqrt{2}$, which is purely periodic with expansion [3, 3, ...]. On the other hand, if $x = [4, 4, 4, ...]$ or $x = [4, x]$, then $x = 4 + \frac{\sqrt{2}}{x}$, and thus $x = 2 + \sqrt{4 + \sqrt{2}}$, which is an expression with nested radicals that cannot be simplified.

Now it is worth giving the definition of **quadratic surd**.

Definition. A number of the form $\sqrt{a}$, where a is a positive rational number which is not the square of another rational number is called a pure quadratic surd. A number of the form $a+\sqrt{b}$, where a is rational and $\sqrt{b}$ is a pure quadratic surd is sometimes called a mixed quadratic surd [2, p. 20]. A quadratic surd is also an irrational number that is the solution to some quadratic equation whose coefficients are integers, and can therefore be expressed as

$$\frac{a + b\sqrt{c}}{d},$$

for integers a, b, c, d; with b, c and d non-zero, and with c square-free.

Theorem 3.1.1. *if x is rational, then every x_k is either rational or an irrational quadratic surd.*

Proof. Assuming that we start with x being rational, by way of induction there are two cases for subsequent values of x:

Case 1: Suppose x_j is rational. If x_j is an integer, we are done. Otherwise, $x_j = u/v$ for some u and v, relatively prime, and, therefore, $x_{j+1} = \frac{\sqrt{n}}{x_j - a_j} = \frac{v\sqrt{n}}{u - a_j v} = \frac{v\sqrt{n}}{d}$ for $d = u - a_j v$, which is a quadratic surd.

Case 2: If x_j is a quadratic surd, then we may assume $x_j = \frac{b+c\sqrt{n}}{d}$ for integers b, c, d, so then $x_{j+1} = \frac{\sqrt{n}}{x_j - a_j} = \frac{\sqrt{n}}{\frac{b+c\sqrt{n}}{d} - a_j} = \frac{d\sqrt{n}}{b - a_j d + c\sqrt{n}}$. Now, If $b - a_j d = 0$, then x_{j+1} is a rational: $x_{j+1} = d/c$. Note that c is not 0 by the assumption that x_j is a surd. On the other hand, if $b - a_j d$ is not 0, we let $e = b - ad$, getting $x_{j+1} = \frac{-cdn + de\sqrt{n}}{e^2 - c^2 n}$. Next, we let $f = -cdn$, $g = de$ and $h = e^2 - c^2 n$ to obtain $x_{j+1} = \frac{f + g\sqrt{n}}{h}$, which is a quadratic surd. ■

As a consequence of Theorem 2.2.2, if a real number x has an eventually periodic expansion, then some x_k will have a purely periodic expansion. Since we are interested in expansions of rational x, by Theorem 3.2.1, this x_k must be a quadratic surd. However, in Example 3.1.1, we showed that not all x_k that give a periodic expansion is a rational or a quadratic surd. This naturally led us to the question

"Which quadratic surds can have purely periodic behavior?". This question is probably too hard to tackle directly. However, we shall show how to answer it for periods of small length, especially lengths 1 and 2, which were the ones that came up the most in Table 2.3.1.

3.2 Period Length 1

We now turn our attention to understanding purely periodic expansions with short periods. If an expansion is purely periodic with period length 1, then it has the following form

$$x = d + \frac{\sqrt{n}}{x}. \tag{3.2.1}$$

We can solve for x to get

$$x = \frac{d + \sqrt{d^2 + 4\sqrt{n}}}{2}. \tag{3.2.2}$$

By Theorem 3.1.1, the x in formula (3.2.2) must be either rational or a quadratic surd. Consequently, the nested radicals must simplify. The only way this can happen is if $d^2 + 4\sqrt{n}$ is a perfect square. Thus, we write

$$d^2 + 4\sqrt{n} = (u + v\sqrt{n})^2,$$

or

$$d^2 + 4\sqrt{n} = u^2 + nv^2 + 2uv\sqrt{n},$$

so we must have

$$u^2 + nv^2 = d^2 \quad \text{and} \quad 2uv = 4.$$

Therefore, we have two cases: first we might have u = 1 and v = 2, which gives

$$d^2 = 1 + 4n. \tag{3.2.3}$$

Alternatively, if u = 2 and v = 1 then

$$d^2 = 4 + n. \tag{3.2.4}$$

Now, we first find solutions to (3.2.3). We need d to be odd, so we set d = 2k+1. Solving for n, we get $n = k^2 + k$. This means that (d, n) has to have the form $(2k + 1, k^2 + k)$ for some positive integer k. Similarly, we find solutions to (3.2.4). However, in this case, d can be odd or even. If d is odd, then we have $d = 2k + 1$. Solving for n, we get $n = 4k^2 + 4k - 3$. This means that (d, n) has to have the form $(2k + 1, 4k^2 + 4k - 3)$ for some positive integer k. On the other hand, if d is even, then we have $d = 2k$. Solving for n, we get $n = 4k^2 - 4$. This means that (d, n) has to have the form $(2k, 4k^2 - 4)$ for some positive integer k. As a result, we have three families of solutions to n.

Theorem 3.2.1. *There are infinitely many quadratic surds that can be expanded into purely periodic expansions with period length 1.*

Proof. We proceed with a direct proof and with the conditions in case (3.2.3). We need $d = 2k+1$ and $d^2 + 4\sqrt{n} = (1+2\sqrt{n})^2$, with $n = k^2 + k$. Thus, we have

$$x = \frac{2k+1+1+2\sqrt{k^2+k}}{2} = k+1+\sqrt{k^2+k}.$$

Now that we have found a description for x, we show it can actually be expanded into a period expansion with period 1. If we run our continued fraction algorithm, we have

$$a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor = \left\lfloor k+1+\sqrt{k^2+k} \right\rfloor.$$

Notice that $\sqrt{k^2+k+k+1} = \sqrt{k^2+2k+1} = \sqrt{(k+1)^2} = k+1$. Hence, $k < \sqrt{k^2+k} < k+1$, and then we can say

$$\left\lfloor k+1+\sqrt{k^2+k} \right\rfloor = \lfloor k+1+k+\lambda \rfloor,$$

where $0 < \lambda < 1$. Therefore,

$$a_0 = \lfloor x_0 \rfloor = \lfloor k+1+k+\lambda \rfloor = 2k+1.$$

For the next step, we have

$$x_1 = \frac{z}{x_0 - a_0} = \frac{\sqrt{k^2+k}}{(k+1+\sqrt{k^2+k})-(2k+1)} = \frac{k^2+k+k\sqrt{k^2+k}}{k^2+k-k^2} = k+1+\sqrt{k^2+k}.$$

Since $x_1 = x_0$, the expansion is purely period with period length 1. Therefore, using our compact continued fraction symbol, for every positive integer k, we have

$$k+1+\sqrt{k^2+k} = [2k+1, 2k+1, 2k+1, ...] = [\overline{2k+1}].$$

For the case in (3.2.4), if d is odd, we need $d = 2k+1$ and $d^2 + 4\sqrt{n} = (2+\sqrt{n})^2$, with $n = 4k^2+4k-3$. Thus, we have

$$x = \frac{2k+3+\sqrt{4k^2+4k-3}}{2} = [2k+1, 2k+1, 2k+1, ...] = [\overline{2k+1}].$$

Here we again show that x can actually be expanded into a period expansion with period 1. If we run our continued fraction algorithm, we have $a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor = \left\lfloor \frac{2k+3+\sqrt{4k^2+4k-3}}{2} \right\rfloor$. Notice that $\sqrt{4k^2+4k-3+4} = \sqrt{4k^2+4k+1} = \sqrt{(2k+1)^2} = 2k+1$, which means $2k < \sqrt{4k^2+4k-3} < 2k+1$, and then we can write

$$a_0 = \left\lfloor \frac{2k+1+2+2k+\lambda}{2} \right\rfloor,$$

where $0 < \lambda < 1$. Thus, we have

$$a_0 = \left\lfloor 2k+1+\frac{1+\lambda}{2} \right\rfloor.$$

Since $0 < \frac{\lambda}{2} < \frac{1}{2}$, we can say

$$a_0 = \left\lfloor 2k+1+\frac{1+\lambda}{2} \right\rfloor = 2k+1.$$

For the next step of the algorithm, we have

$$x_1 = \frac{z}{x_0 - a_0} = \frac{\sqrt{4k^2+4k-3}}{\frac{2k+1+2+\sqrt{4k^2+4k-3}}{2} - (2k+1)} = \frac{(1-2k)\sqrt{4k^2+4k-3} - (4k^2+4k-3)}{2(1-2k)} =$$

$$\frac{\sqrt{4k^2+4k-3}}{2} + \frac{-[(1-2k)^2+4(1-2k)]}{2(1-2k)} = \frac{2k+3+\sqrt{4k^2+4k-3}}{2}.$$

Since $x_1 = x_0$, the expansion is purely period with period length 1. On the other hand, if d is even, we need $d = 2k$ and $d^2 + 4\sqrt{n} = (2+\sqrt{n})^2$, with $n = 4k^2 - 4$. Thus, we have

$$x = k+1+\sqrt{k^2-1} = [2k, 2k, 2k, ...] = [\overline{2k}].$$

We finally demonstrate once again that x can actually be expanded into a period expansion with period 1 in this case as well. If we run our continued fraction algorithm, we have

$$a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor = \left\lfloor k+1+\sqrt{k^2-1} \right\rfloor.$$

Notice that $\sqrt{k^2-2k-1+2} = \sqrt{k^2-2k+1} = \sqrt{(k-1)^2} = k-1$, which means $k-1 < \sqrt{k^2-1} < k$, and so, we can write $a_0 = \lfloor k+1+k-1+\lambda \rfloor$, where $0 < \lambda < 1$. Thus, we have $a_0 = 2k$. For the next step of the algorithm, we have

$$x_1 = \frac{z}{x_0 - a_0} = \frac{2\sqrt{k^2-1}}{k+1+\sqrt{k^2-1}-2k} = \frac{2(1-k)\sqrt{k^2-1} + 2(1-k)(k+1)}{2(1-k)} = k+1+\sqrt{k^2-1}. \tag{3.2.5}$$

Notice that $x_1 = x_0$, which means the expansion is purely period with period length 1. To summarize, the quadratic surds with purely periodic behavior with period 1 are

$$k+1+\sqrt{k^2+k} = [\overline{2k+1}]_{\sqrt{n}}, \text{ with } n = k^2+k \text{ and } k \geq 1, \tag{3.2.6}$$

$$\frac{2k+3+\sqrt{4k^2+4k-3}}{2} = [\overline{2k+1}]_{\sqrt{n}}, \text{ with } n = 4k^2+4k-3 \text{ and } k \geq 1, \tag{3.2.7}$$

and

$$x = k + 1 + \sqrt{k^2 - 1} = [\overline{2k}]_{\sqrt{n}}, \text{ with } n = 4k^2 - 4 \text{ and } k \geq 2. \tag{3.2.8}$$

■

By Theorem 3.2.1, a rational x with an eventually periodic expansion of period 1 has some x_j of the type given in equations (3.2.6), (3.2.7) or (3.2.8). This means that not all values of n can give periodic expansions of period 1, but only those of the forms given in these three equations. However, even among the allowable values of n, we could not find examples for every one of them in our search, as it can be seen next.

We use equations in (3.2.6), (3.2.7) or (3.2.8) to obtain all the allowable values of $n < 100$ and, if applicable, we give an example of a rational x that gives a periodic expansion with period length 1 using such a value of n.

Table 3.2.1: $n = k^2 + k$

n	**example**
2	$x = \frac{9}{5}$
6	$x = \frac{3}{2}$
12	not found
20	not found
30	not found
42	not found
56	not found
72	not found
90	not found

Table 3.2.2: $n = 4k^2 + 4k - 3$

n	**example**
5	not found
21	not found
45	not found
77	not found

Table 3.2.3: $n = 4k^2 - 4$

n	**example**
12	$x = \frac{3}{2}, \frac{8}{7}$
32	$x = \frac{4}{3}, \frac{18}{17}$
60	$x = \frac{5}{4}, \frac{32}{31}$
96	$x = \frac{6}{5}, \frac{50}{49}$

Notice that 12 appears twice: first on Table 3.2.1, and again on Table 3.2.3. The reason for this is because 12 is given by two families of solutions, namely $k^2 + k$ with $k = 3$ and $4k^2 - 4$ with $k = 2$, but they present different periodic behaviors: $4 + \sqrt{12} = [7, 7, 7, ...]$ and $3 + \sqrt{3} = [4, 4, 4, ...]$. However, during our search for periodic behaviors with n=12, the only pattern that showed up was $[a_0, a_1, ..., a_j, 4, 4, 4, ...]$. We could not find a single expansion with period of 7's rather than 4's.

Another interesting point to notice is that, in the previous section, we mentioned that all periodic expansions with period length 1 and with $z = \sqrt{2}$ became periodic with the number 3. It turns out that this result is expected. If we use the equation $n = k^2 + k$ with $k = 1$ then $n = 2$. In this case, $d = 2k + 1 = 3$. Now if we replace both n and d into equation (3.2.6), we get $x = k + 1 + \sqrt{k^2 + k} = 2 + \sqrt{2}$, which, also in accordance with our search results, is the number that caused the periodic expansions $[3, 3, 3, ...]$ when $z = \sqrt{2}$.

Additionally, Table 3.2.3 gives us a very special pattern which helps us connect equation (3.2.8) back to rationals. Notice that $n = 12, 32, 60, 96$ all come from the formula $4k^2 - 4$ with $k = 2, 3, 4, 5$, respectively. The smallest rational that generates an expansion with each of these n are, $3/2, 4/3, 5/4$ and $6/5$, respectively. Furthermore, each expansion becomes periodic when $x = 3 + \sqrt{3}, 4 + \sqrt{8}, 5 + \sqrt{15}$ and $6 + \sqrt{24}$, respectively, and all of them have tail length 3. Therefore, we shall hypothesize that $\frac{k+1}{k}$ generates a periodic expansion with period length 1, $\frac{k+1}{k} = [1, a_1, a_2, 2k, 2k, 2k \dots]$.

Theorem 3.2.2. *Every rational number of the form $\frac{k+1}{k}$, for $k > 1$, has a continued fraction expansion of the form $\frac{k+1}{k} = [1, 2k^2 - 2, 2k - 1, 2k, 2k, 2k, ...]_z$, where $z = \sqrt{4k^2 - 4}$.*

Proof. We proceed with a direct proof by expanding $\frac{k+1}{k}$. First we say $x_0 = x = \frac{k+1}{k}$ and calculate $a_0 = \lfloor \frac{k+1}{k} \rfloor = \lfloor 1 + \frac{1}{k} \rfloor$. Since $k > 1$, $a_0 = 1$. Now, we calculate

$$x_1 = \frac{z}{x_0 - a_0} = \frac{\sqrt{4k^2 - 4}}{\frac{k+1}{k} - 1} = 2k\sqrt{k^2 - 1}.$$

For the second step, we have $a_1 = \lfloor 2k\sqrt{k^2 - 1} \rfloor = \lfloor \sqrt{4k^4 - 4k^2} \rfloor$. Notice that $\sqrt{4k^4 - 4k^2 + 1} = \sqrt{(2k^2 - 1)^2} = 2k^2 - 1$, and that $\sqrt{4k^4 - 4k^2 - 4k^2 + 4} = \sqrt{4k^4 - 8k^2 + 4} = \sqrt{(2k^2 - 2)^2} = 2k^2 - 2$, which means $2k^2 - 2 < \sqrt{4k^4 - 4k^2} < 2k^2 - 1$. Hence, $a_1 = 2k^2 - 2$. Next, we calculate

$$x_2 = \frac{z}{x_1 - a_1} = \frac{\sqrt{4k^2 - 4}}{(2k\sqrt{k^2 - 1}) - (2k^2 - 2)} = \frac{\sqrt{k^2 - 1}}{1 - k^2 + k\sqrt{-1 + k^2}} = k + \sqrt{k^2 - 1}.$$

Please, refer to (3.2.5) for the calculations of the third step. We have

$$x_3 = k + 1 + \sqrt{k^2 - 1}.$$

Notice that the above is equal to the result in (3.2.8), which means $a_k = 2k$ for all $k > 2$, and therefore, the expasion of $\frac{k+1}{k}$ is purely periodic with period length 1. ∎

Similarly, Table 3.2.3 gives us another pattern: for each n=12, 32, 60 and 96, we found another fraction with periodic expansion, namely, $\frac{8}{7}, \frac{18}{17}, \frac{32}{31}, \frac{50}{49}$, respectively. Thus, we have the following theorem:

Theorem 3.2.3. *Every rational number of the form $\frac{2k^2}{2k^2-1}$, for $k > 1$, has a continued fraction expansion of the form $\frac{2k^2}{2k^2-1} = [1, 4k^3 - 4k, 4k^2 - 3, 2k - 1, 2k, 2k, 2k, ...]_z$, where $z = \sqrt{4k^2 - 4}$.*

Proof. We proceed with a direct proof by expanding $\frac{2k^2}{2k^2-1}$. First we say $x_0 = x = \frac{2k^2}{2k^2-1} = \frac{1}{1-\frac{1}{2k^2}}$. Since $k > 1$, $2k^2 \geq 8$, and so $1 < x_0 < 2$. This means $a_0 = \lfloor x_0 \rfloor = 1$ for all $k > 1$. Now, we calculate $x_1 = \frac{z}{x_0 - a_0} = (2k^2 - 1)\sqrt{4k^2 - 4} = \sqrt{16k^6 - 32k^4 + 20k^2 - 4}$. Notice that $16k^6 - 32k^4 + 16k^2 = (4k^3 - 4k)^2$ and $16k^6 - 32k^4 + 8k^3 + 16k^2 - 8k + 1 = (4k^3 - 4k + 1)^2$, which means $4k^3 - 4k < \sqrt{16k^6 - 32k^4 + 20k^2 - 4} < 4k^3 - 4k + 1$, and so $a_1 = \lfloor x_1 \rfloor = 4k^3 - 4k$. After some algebra, we have $x_2 = \frac{z}{x_1 - a_1} = \sqrt{4k^4 - 4k^2} + 2k^2 - 1$. Since $2k^2 - 2 < \sqrt{4k^4 - 4k^2} + 2k^2 - 1 < 2k^2 - 1$, $a_2 = \lfloor x_2 \rfloor = 2k^2 - 2 + 2k^2 - 1 = 4k^2 - 3$. Then, $x_3 = \frac{z}{x_2 - a_2} = k + \sqrt{k^2 - 1}$. Since $k - 1 < \sqrt{k^2 - 1} < k, a_3 = \lfloor x_3 \rfloor = k - 1 + k = 2k - 1$. Once again,

$x_4 = \frac{z}{x_3 - a_3}$, which, after some algebra, gives $x_4 = k+1+\sqrt{k^2-1}$. Thus, $a_4 = \lfloor x_4 \rfloor = k-1+k+1 = 2k$. Finally, $x_5 = \frac{z}{x_4 - a_4} = \frac{\sqrt{-4+4k^2}}{1-k+\sqrt{-1+k^2}} = k+1+\sqrt{k^2-1}$. Since $x_5 = x_4$, $a_k = 2k$ for all $k > 3$, and therefore, the expasion of $\frac{2k^2}{2k^2-1}$ is purely periodic with period length 1. ■

3.3 Period Length 2

If x is purely periodic with period 2, then x has an expansion of the form $[d, e, d, e, ...]$ and so, by Theorem 2.1.2, $x = x_2$, which means

$$x = d + \frac{\sqrt{n}}{e + \frac{\sqrt{n}}{x}}. \tag{3.3.1}$$

We can solve for x to get

$$x = \frac{de + \sqrt{d^2e^2 + 4de\sqrt{n}}}{2e}. \tag{3.3.2}$$

Most of the time, the expression in (3.3.2) will not be a quadratic surd. For example, if $d = 3, e = 4, n = 5$, then $x = \frac{3+\sqrt{9+3\sqrt{5}}}{4}$. However, we are interested in an x which represents an x_j in the expansion of a rational number. This means we want the expression in (3.3.2) to be a quadratic surd. The only way this can happen is if $d^2e^2 + 4de\sqrt{n}$ is a perfect square. Thus, we write

$$d^2e^2 + 4de\sqrt{n} = (u + v\sqrt{n})^2,$$

or

$$d^2e^2 + 4de\sqrt{n} = u^2 + nv^2 + 2uv\sqrt{n},$$

so we must have

$$u^2 + nv^2 = d^2e^2 \quad \text{and} \quad uv = 2de.$$

This means $de = uv/2$, and so

$$u^2 + nv^2 = \frac{u^2v^2}{4},$$

or

$$u^2 = 4n + \frac{16n}{v^2 - 4}. \tag{3.3.3}$$

Theorem 3.3.1. *For any given n, there are at most finitely many purely periodic quadratic surds of period 2.*

Proof. As we derived above, we need $u^2 = 4n + \frac{16n}{v^2-4}$. Notice that $v^2 - 4$ must be at most as large as $16n$, which means only a finite number of v can work, and for each v, at most one u. Given a u and a v, by (3.3.2), we must have $x = \frac{de+u+v\sqrt{n}}{2e}$. ■

For $n < 100$, there are 26 allowable values of n, that is to say, numbers n for which we could find solutions u and v in (3.3.3). The following is a table displaying such values of n and their respective u, v, d, e and x solutions. Notice that, for each pair of solutions u and v, one can always get the permutations (d, e) and (e, d). However, if $x = x_0$ has periodic expansion $[\overline{d, e}]$, then so does every x_k with k even, and every x_k with k odd has expansion $[\overline{e, d}]$, which means it is not necessary to have both (d, e) and (e, d). For this reason, we only list solutions for which $e \leq d$, though when d = e, the resulting x has period 1 rather than period 2. Moreover, we are only interested in maximal expansions, so, by Theorem 2.1.3, we rule out solutions that do not have $\lfloor\sqrt{n}\rfloor \leq e \leq d$.

Table 3.3.1: Allowable values of n in equation (3.3.3)

n	**u**	**v**	**list of (d, e) and x values**
2	3	6	(3,3) $x = 2 + \sqrt{2}$, (9,1)* $x = 6 + 4\sqrt{2}$
3	4	4	(4,2) $x = 3 + \sqrt{3}$, (8,1)* $x = 6 + 2\sqrt{3}$
5	6	3	(3,3) $x = \frac{5+\sqrt{5}}{2}$
6	5	10	(5,5) $x = 3 + \sqrt{6}$
8	6	6	(6,3) $x = 4 + \sqrt{8}$, (9,2)* $x = 3(2 + \sqrt{2})$
12	8	4	(4,4) $x = \frac{6+\sqrt{12}}{2}$
12	7	14	(7,7) $x = 4 + \sqrt{12}$
15	8	8	(8,4) $x = 5 + \sqrt{15}$
20	9	18	(9,9) $x = 5 + \sqrt{20}$
21	10	5	(5,5) $x = \frac{7+\sqrt{21}}{2}$
24	10	10	(10,5) $x = 6 + \sqrt{24}$
30	11	22	(11,11) $x = 6 + \sqrt{30}$
32	12	6	(6,6) $x = \frac{8+\sqrt{32}}{2}$
35	12	12	(9,8) $x = \frac{21+3\sqrt{35}}{4}$, (12,6) $x = 7 + \sqrt{35}$
42	13	26	(13,13) $x = 7 + \sqrt{42}$
45	14	7	(7,7) $x = \frac{9+\sqrt{45}}{2}$
48	14	14	(14,7) $x = 8 + \sqrt{48}$
56	15	30	(15,15) $x = 8 + \sqrt{56}$, (25,9) $x = \frac{40+5\sqrt{56}}{3}$
60	16	8	(8,8) $x = \frac{10+\sqrt{60}}{2}$
63	16	16	(16,8) $x = 9 + \sqrt{63}$
72	17	34	(17,17) $x = 9 + \sqrt{72}$
77	18	9	(9,9) $x = \frac{11+\sqrt{77}}{2}$
80	18	18	(18,9) $x = 10 + \sqrt{80}$
90	19	38	(19,19) $x = 10 + \sqrt{90}$
96	20	10	(10,10) $x = \frac{12+\sqrt{96}}{2}$
99	20	20	(20,10) $x = 11 + \sqrt{99}$

Remark. the asterisk, *, indicates that the x does not have the desired maximal expansion.

However, not all values of n in Table 3.3.1 lead to a quadratic surd with period 2. For example, with

$n = 6$, we have $2de = uv = 50$, so $de = 25$. This has three solutions $(d, e) : (1, 25), (5, 5), (25, 1)$. If $d = 5 = e$, then we have period 1, not period 2. Furthermore, the other two cases cannot happen with the maximal expansion, because, by Theorem 2.1.3, we need d and e to be at least as large as $\lfloor \sqrt{5} \rfloor = 2$, so we cannot have $d = 1$ or $e = 1$. Thus, it is not possible to obtain any periodic behavior with length 2 from $n = 6$.

Even if n has a quadratic surd with period 2, we are not guaranteed to be able to find a rational x that leads to a periodic expansion with period 2. For example, the quadratic surds marked with an asterisk: $x = 6 + 4\sqrt{2}$ and $x = 6 + 2\sqrt{3}$, for $n = 2$ and $n = 3$, respectively, give unexpected expansions. The former has expansion $[10, 5, 1, 2, 2]$ and latter has expansion $[9, 3, 2, 4, 2, 4, ...]$. However, they were supposed to give purely period expansions with periods $[9, 1]$ and $[8, 1]$, respectively. While the latter does have a periodic expansion, it is not pure and does not have period $[8, 1]$, and the former is not even periodic.

According to our search, among all the values of n on Table 3.3.1, only the following actually have rationals whose expansion is periodic with period length 2: $3, 8, 15, 24, 35, 48, 56, 63, 80, 99$. Notice that, except for 56, all these values match the values of n for period lenth 2 on Table 2.3.1. If we take a closer look, we will see that these numbers follow the pattern described by $n = k^2 - 1$ for some $k > 1$. Also, the smallest rationals that lead to period 2 with each of these values of n follow a pattern similar to the one described by Theorem 3.2.2.

Theorem 3.3.2. *There are infinitely many rationals $\frac{k+1}{k}$ that have eventual periodic expansion with period length 2 and $n = k^2 - 1$, for some $k > 1$.*

Proof. We proceed with a direct proof by expanding $\frac{k+1}{k}$. By Theorem 3.2.2, we know $a_0 = 1$. So, we calculate

$$x_1 = \frac{z}{x_0 - a_0} = \frac{\sqrt{k^2 - 1}}{\frac{k+1}{k} - 1} = k\sqrt{k^2 - 1}.$$

For the second step, we have $a_1 = \lfloor k\sqrt{k^2 - 1} \rfloor = \lfloor \sqrt{k^4 - k^2} \rfloor$. Notice that $\sqrt{k^4 - 2k^2 + 1} = \sqrt{(k^2 - 1)^2} = k^2 - 1$, which means $k^2 - 1 < \sqrt{k^4 - k^2} < k^2$. Hence, $a_1 = k^2 - 1$. Now, we calculate

$$x_2 = \frac{z}{x_1 - a_1} = \frac{\sqrt{k^2 - 1}}{k\sqrt{k^2 - 1} - (k^2 - 1)} = k + \sqrt{k^2 - 1}.$$

Please, refer to (3.2.5) for complete explanation for the next steps. We know $k - 1 < \sqrt{k^2 - 1} < k$, thus $a_2 = \lfloor k + \sqrt{k^2 - 1} \rfloor = 2k - 1$. So, we have

$$x_3 = \frac{z}{x_2 - a_2} = \frac{\sqrt{k^2 - 1}}{(k + \sqrt{k^2 - 1}) - (2k - 1)} = \frac{k + 1 + \sqrt{k^2 - 1}}{2}.$$

For the forth step, we have $a_3 = \left\lfloor \frac{k+1+\sqrt{k^2-1}}{2} \right\rfloor = k$. Then,

$$x_4 = \frac{z}{x_3 - a_3} = \frac{\sqrt{k^2-1}}{\frac{k+1+\sqrt{k^2-1}}{2} - k} = k+1+\sqrt{k^2-1}.$$

Finally, $a_4 = \left\lfloor k+1+\sqrt{k^2-1} \right\rfloor = 2k$, so

$$x_5 = \frac{z}{x_4 - a_4} = \frac{\sqrt{k^2-1}}{k+1+\sqrt{k^2-1}-2k} = \frac{k+1+\sqrt{k^2-1}}{2}.$$

Notice that $x_5 = x_3$, thus $\frac{k+1}{k}$ has a periodic expansion with period length 2, and its expansion has the form $\frac{k+1}{k} = [1, k^2-1, 2k-1, k, 2k, k, 2k, k, 2k, ...] = [1, k^2-1, 2k-1, \overline{k, 2k}]$. ■

3.4 Longer Periods

In this last section, we give a brief discussion on period length 3 and 4. If x is purely periodic with period 3, then x has an expansion of the form $[d, e, f, d, e, f, ...]$ and so, by Theorem 2.1.2, $x = x_3$, which means

$$x = d + \frac{\sqrt{n}}{e + \frac{\sqrt{n}}{f + \frac{\sqrt{n}}{x}}}.$$

We can solve for x to get

$$x = \frac{def + d\sqrt{n} - e\sqrt{n} + f\sqrt{n} + \sqrt{d^2e^2f^2 + 2d^2ef\sqrt{n} + 2de^2f\sqrt{n} + 2def^2\sqrt{n} + d^2n + 2den + e^2n + 2dfn + 2efn + f^2n + 4n^{3/2}}}{2ef + 2\sqrt{n}}.$$

Since we want the above equation to be a quadratic surd, the number inside the radical must be a perfect square. Thus, we write

$$d^2e^2f^2 + 2d^2ef\sqrt{n} + 2de^2f\sqrt{n} + 2def^2\sqrt{n} + d^2n + 2den + e^2n + 2dfn + 2efn + f^2n + 4n^{3/2} = (u + v\sqrt{n})^2$$

or

$$d^2e^2f^2 + 2d^2ef\sqrt{n} + 2de^2f\sqrt{n} + 2def^2\sqrt{n} + d^2n + 2den + e^2n + 2dfn + 2efn + f^2n + 4n^{3/2} = u^2 + nv^2 + 2uv\sqrt{n},$$

which means, after some algebra, we must have

$$u^2 + nv^2 = d^2e^2f^2 + n(d+e+f)^2 \tag{3.4.1}$$

and

$$uv = 2n + def(d+e+f). \tag{3.4.2}$$

Similarly, if x is purely periodic with period 4, then x has an expansion of the form $[d, e, f, g, d, e, f, g, ...]$ and so, by Theorem 2.1.2, $x = x_4$, which means

$$x = d + \cfrac{\sqrt{n}}{e + \cfrac{\sqrt{n}}{f + \cfrac{\sqrt{n}}{g + \cfrac{\sqrt{n}}{x}}}}.$$

We can solve for x to get

$$x = \frac{defg + de\sqrt{n} - ef\sqrt{n} + dg\sqrt{n} + fg\sqrt{n} + \sqrt{\lambda}}{2(e(fg + \sqrt{n}) + g\sqrt{n})},$$

where

$$\lambda = d^2e^2f^2g^2 + 2d^2e^2fg\sqrt{n} + 2de^2f^2g\sqrt{n} + 2d^2efg^2\sqrt{n} + 2def^2g^2\sqrt{n} + d^2e^2n + 2de^2fn + e^2f^2n+$$

$$2d^2egn + 8defgn + 2ef^2gn + d^2g^2n + 2dfg^2n + f^2g^2n + 4den^{3/2} + 4efn^{3/2} + 4dgn^{3/2} + 4fgn^{3/2}.$$

Since we want x to be a quadratic surd, the number inside the radical λ must be a perfect square. Thus, we write

$$\lambda = (u + v\sqrt{n})^2$$

or

$$\lambda = u^2 + nv^2 + 2uv\sqrt{n},$$

which means, after some algebra, we must have

$$u^2 + nv^2 = d^2e^2f^2g^2 + n((d + f)^2(e + g)^2 + 4defg), \tag{3.4.3}$$

$$uv = (d + e)(f + g)(defg + 2n). \tag{3.4.4}$$

We proceeded with a numerical method to try to find simultaneous solutions to (3.4.1) and (3.4.1) as well as (3.4.3) and (3.4.4). We let n range between 2 and 1000 (excluding those that are perfect squares). Since we only wanted maximal expansions, we had d go from $\lfloor\sqrt{n}\rfloor$ to 100. Moreover, we wanted $d < e < f$, so we had e go from d to 100 and f go from e + 1 to 100. In each case, we checked if there were integers $u, v > 0$ that satisfied the equations. However, no solutions turned up other than those found in our data.

4 Other Results

4.1 Finite Expansions

When doing expansions of various rationals for various values of n, we noticed that the shortest possible finite expansion appeared to have length 4. We first show that the shortest expansion a rational can have has indeed length 4. We then prove a pattern that connects finite expansions of this length with values of n that lead to these expansions. Please, remember that if two rationals differ only by an integer, then they have the same expansion, except for the first term. Thus, for simplification, in the following two theorems, we assume $\lfloor x \rfloor = 0$.

Theorem 4.1.1. *The shortest expansion a rational can have has length 4.*

Proof. We assume x is a rational, so, by Theorem 3.1.1, x_1 is a quadratic surd, which means $a_1 = \lfloor x_1 \rfloor$, and thus, x has expansion with length at least 2. If x has expansion with length 2, then $x = [a, b] = [0, b] = \frac{\sqrt{n}}{b}$, but we assumed x is a rational, so it cannot have expansion with length 2. Finally, if x has expansion with length 3, then

$$x = \frac{\sqrt{n}}{b + \frac{\sqrt{n}}{c}} = \frac{c\sqrt{n}}{bc + \sqrt{n}}.$$

We multiply both sides by the denominator to obtain $xbc + x\sqrt{n} = c\sqrt{n}$ or $xbc + (x - c)\sqrt{n} = s + t\sqrt{n} = 0$, for $s = xbc$ and $t = x - c$. This can only happen if $s = 0$ and $t = 0$. Thus, we have $x - c = 0$, which means $x = c$. However, x must be a rational, so it cannot have expansion with length 3 either. Since x cannot have expansion with length 1, 2 or 3, but can have expansion with length 4, the shortest expansion x can have has length 4. ∎

Theorem 4.1.2. *If x is a rational and has a finite expansion $[a, b, c, d]_{\sqrt{n}}$, then $n = \frac{bc^2d^2}{b+d}$.*

Proof. Since we assume $\lfloor x \rfloor = 0$, we can say $x = [0, b, c, d]$, and thus

$$x = \frac{\sqrt{n}}{b + \frac{\sqrt{n}}{c + \frac{\sqrt{n}}{d}}} = \frac{n + cd\sqrt{n}}{bcd + (b + d)\sqrt{n}}.$$

Now, we multiply both sides by the denominator

$$x(bcd + (b + d)\sqrt{n}) = \frac{n + cd\sqrt{n}}{bcd + (b + d)\sqrt{n}}$$

or

$$xbcd - n + (xb + xd - cd)\sqrt{n} = s + t\sqrt{n} = 0,$$

for $s = xbcd - n$ and $t = xb + xd - cd$. This can only happen if $s = 0$ and $t = 0$. Thus, we have

$xb + xd - cd = 0$, which means $x = \frac{cd}{b+d}$. We also have $xbcd - n = 0$, which means $x = \frac{n}{bcd}$. Hence,

$$\frac{n}{bcd} = \frac{cd}{b+d},$$

and therefore,

$$n = \frac{bc^2d^2}{b+d}.$$

■

Theorem 4.1.3. *For rational numbers between 1 and 2, there are exactly two maximal expansions of length* 4*, namely* $3/2 = [1, 2, 1, 2]$ *when* $n = 2$ *and* $5/4 = [1, 6, 1, 2]$ *when* $n = 3$.

Proof. For this proof, we use the fact that $\frac{xy}{x+y} \geq \frac{min(x,y)}{2}$, whose proof can be found in the appendix. Let $\lfloor\sqrt{n}\rfloor = k$, so $k^2 < n < (k+1)^2$. By Theorem 2.1.3, each of b, c, d must be at least as large as k, so $n = \frac{bc^2d^2}{b+d} \geq c^2d\frac{min(b,d)}{2} \geq k^2k\frac{k}{2} = \frac{k^4}{2}$. Since $n < (k+1)^2$, $n \leq k^2 + 2k$. Thus, $k^2 + 2k \geq \frac{k^4}{2}$ or $4 \geq k(k^2 - 2)$, which forces $k \leq 2$. If $k = 2$, then $n \leq k^2 + 2k = 8$ and $n \geq \frac{k^4}{2} = 8$, and so we can only have $n = 8$. However, we know that $n \geq c^2d\frac{min(b,d)}{2}$, and that when an expansion terminates, the last partial quotient d must be greater than z. This means we need $d > \sqrt{8}$ or $d > 2$. We also know that $b, c \geq 2$. Thus, $c^2d\frac{min(b,d)}{2} \geq 2 \cdot 3 \cdot 2 = 12 > 8$, a contradiction, ruling out all possible solutions. Now, when $k = 1$, $n \leq k^2 + 2k = 3$ and $n \geq \frac{k^4}{2} = \frac{1}{2}$, and so we can only have $n = 2$ or $n = 3$. If $n = 2$, then $d \geq 2$, and $2 \geq c^2d\frac{min(b,d)}{2}$, which forces $c = 1$. Thus, we have $n = \frac{bd^2}{b+d}$ or $nd = b(d^2 - n)$. This means $d^2 - n$ must divide nd, which can only happen when $d = 2$, because, if $d \geq 3$, then $d^2 - 2 > 2d$. Hence, for $n = 2$, there is only one solution: $b = 2$, $c = 1$, $d = 2$ and

$$x = \frac{\sqrt{2}}{2 + \frac{\sqrt{2}}{1+\frac{\sqrt{2}}{2}}} = \frac{2+\sqrt{2}}{2\sqrt{2}(1+\sqrt{2})} = \frac{1}{2}.$$

However, we want rationals between 1 and 2, so $1 + \frac{1}{2} = \frac{3}{2} = [1, 2, 1, 2]_{\sqrt{2}}$. Similarly, if $n = 3$, then $d \geq 2$, and $3 \geq c^2d\frac{min(b,d)}{2}$, which forces $c = 1$. Again, $d^2 - n$ must divide nd, which also can only happen when $d = 2$, because, when $d = 3$, $d^2 - 3 = 6$ does not divide $3d$, and if $d \geq 4$, then $d^2 - 3 > 3d$. Therefore, for $n = 3$, there is only one solution: $b = 6$, $c = 1$, $d = 2$ and

$$x = \frac{\sqrt{3}}{6 + \frac{\sqrt{3}}{1+\frac{\sqrt{3}}{2}}} = \frac{\sqrt{3}(2+\sqrt{3})}{4(3+2\sqrt{3})} = \frac{1}{4}.$$

Since, we want rationals between 1 and 2, $1 + \frac{1}{4} = \frac{5}{4} = [1, 6, 1, 2]_{\sqrt{3}}$. ■

5 Future Work

From the beginning, there were several questions regarding continued fractions, and more have been raised during this project. We started off with a simple, but nice algorithm to expand rationals into continued fractions. Althought we did not expand a staggering number of rationals, and each expansion was not as deep as desired, we were still able to use these expansions to develop a theory on non-simple continued fractions. However, we belive that it might be nice if one could improve our algorithm or come up with some algorithm more efficient that could extend the calculations deeper, that is, to more rationals (larger denominators), and to greater depth (we did expansions with a depth of 500 for a few values of n, a depth of 200 for some values of n and a depth of 100 for most values of n).

Other possibilities for future work include:

(1) With regard to $n = \sqrt{2}$, one could prove why roughly 72% of rationals have a terminating expansion, and one could also investigate expansions of quadratic surds rather than rationals to verify if it is still 72%.

(2) Find efficient ways to backtrack. That is to say, given some quadratic surd, y, one could find ways to backtrack to a rational r so that y occurs among the values of x when r is expanded.

(3) It would certainly be fruitful to work on larger period lengths, especially lengths 3 and 4, trying to find solutions to their equations as well as proving other patterns.

(4) Our investigation on finite expansions led us to conjecture that for any finite length, $r = [a_0, a_1, ..., a_m]$, there are only finitely many n for which r has such an expansion, and only finitely many $a_1, a_2, ..., a_m$ as well.

(5) One could have similar questions about tails where r becomes eventually periodic. For example, is the shortest possible tail 2? Also, for any given fixed tail length, are there finitely many rationals that have expansions with such tail length.

To conclude, there is undoubtedly further investigations to be done on the subject of continued fractions. Many interesting patterns and properties have been uncovered for when numerators are radicals, but inquiries still remain. These discoveries could be useful on other fields of mathematics, and that is an interesting prospect for the future.

References

[1] M. Anselm and S. Weintraub, *A generalization of continued fractions*, J. Number Theory 131 (2011), 2442-2460.

[2] G. Hardy, *A Course in Pure Mathematics*, 10th ed. Cambridge University Press, Cambridge, 1967.

[3] C. Olds, *Continued Fractions*, Random House, New York, 1963.

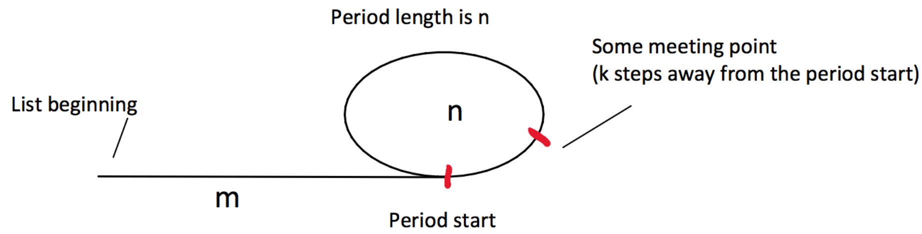
[4] J. Schmieg, *Patterns of Non-Simple Continued Fractions*, UROP project, 2015.

A Proofs of Theorems and Algorithms

A.1 Period Detection: Floyd's Cycle-Finding Algorithm

In order for us to detect periods in a list of numbers, we use the Floyd's Cycle-Finding Algorithm. However, we should not use the list $[a_0, a_1, a_2, ...]$, rather we should use the list $[x_0, x_1, x_2, ...]$, as defined in section 2.1. The reason for this is because we know that if $x_j = x_i$ for $i < j$, then the expansion of x is periodic.

Suppose we are trying to detect a period in the list $[x_0, x_1, x_2, ...]$. The Floyd's Cycle-Finding Algorithm works as follows: (refer to the image below):



(a) List with a period

We have a period with length n and two pointers, the "hare" and the "tortoise", which are initially m steps away from the period. We shall hypothesize that if we move the tortoise 1 step at a time, and the hare 2 steps at a time, they will eventually meet after a total of i steps. Therefore, the following two conditions must hold:

$$i = m + pn + k. \tag{A.1.1}$$

$$2i = m + qn + k. \tag{A.1.2}$$

A.1.1 says that tortoise moves a total of i steps, m steps of which are necessary to get to the beginning of the period. Then, it goes through the period p times for some positive number p before it finally meets hare at k more steps ahead the beginning of the period. Similarly, A.1.2 says that the hare moves a total of 2i steps, m steps of which are necessary to get to the beginning of the period. Then, it goes through the period q times for some positive number q before it finally meets the tortoise at k more steps ahead the beginning of the period.

Now, if we can show that there is at least one set of values for k, q, p that makes A.1.1 and A.1.2 equal, we show that the hypothesis is true. One such set of solutions is $p = 0$, $q = m$ and $k = mn - m$.

We verify that these values work:

$$i = m + pn + k = m + (mn - m) = mn$$

and

$$2i = m + qn + k = m + mn + mn - m = 2mn$$

It is important to notice that this is not necessarily the smallest i possible. In other words, the tortoise and the hare might have already met before many times. However, since we have showed that they meet at some point at least once, we can say that the hypothesis is true. Thus, they would have to meet if we move one of them 1 step, and the other one 2 steps at a time.

The second part of the algorithm consists of finding the beginning of the period and its length. We can do this by first noticing that when the tortoise and the hare meet, the values that they point to are equal. Then we can backtrack by moving each of them one step back and comparing their values, until they are not equal anymore or we reach the beginning of the list. Once either of these happens, we are guaranteed to have found the beginning of the period. The next step is to keep track of this value (we can call it the head of the period) and go through the list of x's again but starting from the head. Once we find another value that matches the head, we are guaranteed to have found the end of the period. Since we know the beginning and the end of the period, we can easily calculate its length.

A.2 Additional Theorems

Theorem A.2.1. *If x and y are positive intergers, then* $\frac{xy}{x+y} \geq \frac{min(x,y)}{2}$.

Proof. This expression is symmetric in x and y, so without loss of generality, we assume $x \geq y$. Then, we have

$$\frac{xy}{x+y} - \frac{y}{2} = \frac{2xy - y(x+y)}{2(x+y)} = \frac{xy - y^2}{2x + 2y}.$$

If $y = x$, then the above expression is equal to 0, otherwise it is nonnegative. ■

B Data Collection

Here we provide a number of tables displaying all rationals that led to periodic or finite expations within the bounds of our search, as described in section 2.3, and, if applicable, we also provide their respective tails and period lengths. Also, notice that the rationals for $n = 2, 3, 6, 8, 12$ are not included on this paper, because there are too many rationals for each of these values of n. However, we provide external files with such data.

B.1 Finite Expansions

Table B.1.1: Rationals that led to finite expansions

n	**rational**	**expansion length**
5	38/21	7
5	26/25	9
5	45/44	6
5	85/44	14
5	53/48	12
5	69/52	38
5	77/73	15
5	75/74	6
5	179/92	5
5	256/129	5
5	160/153	7
5	248/163	17
5	248/165	5
5	234/185	5
5	207/188	8
5	197/192	10
5	204/199	25
5	334/213	45
5	35/239	78
5	278/245	33
5	392/277	27
5	409/295	35
5	331/316	32
5	415/318	36
5	447/370	14
5	517/373	37
7	45/43	5

B.2 Periodic Expansions

Table B.2.1: Rationals that led to peridic expansions

n	rational	tail length	period length
5	344/321	11	4
5	521/356	24	4
10	4/3	2	22
10	13/8	18	22
10	33/29	11	22
10	46/31	10	22
10	47/46	6	22
15	5/4	3	2
24	6/5	3	2
24	25/18	12	2
32	4/3	3	1
32	18/17	4	1
35	7/6	3	2
48	8/7	3	2
60	5/4	3	1
60	32/31	4	1
63	9/8	3	2
80	10/9	3	2
96	6/5	3	1
96	50/49	4	1
99	11/10	3	2

C Mathematica Code

The following code can be pasted as seen and executed in a Mathematica notebook. There is only one function which encompasses the computations needed for this work, which is described in more detail using code comments.

ExpandNumberAsContinuedFraction: this function expands a given rational x into continued fractions using the continued fraction algorithm, as described in section 2.1. It takes three parameters: x, the number to be expanded, a fixed number z and n, which, here, defines the maximum number of terms the expansion can have or the maximum number of steps taken by the algorithm. The function can return three kinds of outputs: {0,0}, if the expansion is finite, {tail, period length} if the expansion is periodic or {-1,-1}, if the algorithm could not define if the expansion is periodic or finite.

```
PopulateExpansionList[limit_] :=
(
   For[i = 0, i < limit, i++,
        (* Maximum number of digits *)
        upperBound = 100000;

         a = Floor[N[x, 5]];
         AppendTo[list, a];
         AppendTo[listx, x];

         (* Finite expasion, so quit! *)
        If[x - a == 0 , Return[{0, 0}],

        (* quit if coefficients are getting too big *)
        If[IntegerLength[IntegerPart[Numerator[x]]] > upperBound ||
        IntegerLength[IntegerPart[Denominator[x]]] > upperBound,
               Return[{-1, -1}];
          ,
                x = Rationalize[Simplify[z/(x - a)]];
          ];
     ];
  ];
);

ExpandNumberAsContinuedFraction[] :=
(
     PopulateExpansionList[4];
    (* Tortoise and Hare initial points *)
     t = 2;
     h = 3;

    While[t <= n/2 ,
    (
       (* Check if the hare and the tortoise have met *)
       If[listx[[t]] == listx[[h]],
       (
            (* Backtrack all the way to find the period beginning *)
          While[listx[[t - 1]] == listx[[h - 1]] && t > 1,
             t--;
             h--;
           ];

          head = t;
          t++;
```

```
            length = 1;

            While[listx[[t]] != listx[[head]],
              length++;
              t++;
            ];

            head--;
            Return[{head, length}];
          )];

            t++;
            h += 2;

            (* Populate expansion list if needed *)
            If[h >= (Length[listx]),
            (
                resultSet = PopulateExpansionList[2];
                Switch[ resultSet, {0, 0}, Return[{0, 0}], {-1, -1},
                Return[{-1, -1}] ];
            )
            ];
      )];
);

(* list of values of a *)
list = {};
(* list of values of x *)
listx = {};
x =  3/2;
z = Sqrt[2];
(* n, here, defines the maximum number of steps taken by the algorithm *)
n = 100;
(* execute the function *)
ExpandNumberAsContinuedFraction[]
(* print the list of values of a *)
list
```