THE BURGSTAHLER COINCIDENCE

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1. INTRODUCTION

Let $\tan x = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$. We know, of course, that $a_{2n} = 0$ for all n. Define a sequence A_n via

$$A_n = \sum_{k=0}^{n-1}\binom{n}{k}a_{k+1} = \sum_{2k<n}\binom{n}{2k}a_{2k+1}. \tag{1.1}$$

We have the following table:

n	1	2	3	4	5	6	7	8	9	10
A_n	1	1	2	3	5	8	$13\frac{2}{45}$	$21\frac{8}{45}$	$34\frac{167}{315}$	$56\frac{20}{63}$

At the 1999 MAA North Central Section Summer Seminar Sylvan Burgstahler posed the following question.

Question 1: Why is A_n approximately equal to a Fibonacci number?

In discussing this problem with the author, Dr. Burgstahler posed two more questions.

Question 2: If a_7 is changed from $\frac{17}{315}$ to $\frac{15}{315} = \frac{1}{21}$, then A_7 becomes 13 and A_8 becomes 21, but the new A_9 is $33\frac{314}{315}$ rather than 34. If we then change a_9 from $\frac{62}{2835}$ to $\frac{63}{2835} = \frac{1}{45}$, A_9 and A_{10} change to the appropriate Fibonacci numbers, but A_{11} remains incorrect. Does this pattern of obtaining two additional Fibonacci numbers for each correction persist?

More generally,

Question 3: Suppose that $f(x) = b_1x + b_3x^3 + b_5x^5 + \cdots$ is such that

$$F_n = \sum_{2k<n}\binom{n}{2k}b_{2k+1},$$

what can be said about the b's, and what can be said about $f(x)$?

In this paper we attempt to answer these questions. The first is straightforward, but the second and third are more interesting. The structure of this paper is as follows. In Section 2 we derive a formula for A_n that explains its proximity to the Fibonacci numbers. In Section 3 we recast this problem as a summation inversion problem to answer Question 2 and part of Question 3. We address the rest of Question 3 in Section 4. Throughout this paper we use the convention that $F_0 = 0$, $F_1 = 1$, α is the golden ratio,

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

We will make free use of the usual facts, e.g., $\alpha+\beta = 1$, $\alpha\beta = -1$.

2. A FORMULA FOR THE NUMBERS A_n

It is well known ([1], formula 4, p. 51) that the coefficients of $\tan x$ can be written explicitly in terms of Bernoulli numbers:

$$a_{2n-1} = (-1)^{n-1}\frac{2^{2n}(2^{2n}-1)}{(2n)!}B_{2n}, \tag{2.1}$$

where B_{2n} is the $2n^{\text{th}}$ Bernoulli number. The Bernoulli numbers are defined by the generating function ([1], formula 1, p. 35)

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \tag{2.2}$$

and have values $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, \ldots$. They satisfy many identities including the recurrence ([1], formula 18, p. 38)

$$\sum_{k=0}^{n-1}\binom{n}{k}B_k = 0,$$

and series formulas

$$B_{2n} = (-1)^{n-1}\frac{(2n)!}{2^{2n-1}\pi^{2n}}\left(1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\cdots\right), \tag{2.3}$$

$$B_{2n} = (-1)^{n-1}\frac{2(2n)!}{(2^{2n}-1)\pi^{2n}}\left(1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\cdots\right). \tag{2.4}$$

These last two formulas can be found in most books of mathematical tables. Alternatively, (2.3) can be found in [1] (formula 22, p. 38) or in [2] (Vol. II, formula 2.60, p. 60). It is easy to derive (2.4) from (2.3).

Using (2.1) and (2.4) with (1.1), we have

$$\begin{aligned} A_n &= \sum_{2k<n}\binom{n}{2k}a_{2k+1} = \sum_{2k<n}\binom{n}{2k}(-1)^k\frac{2^{2k+2}(2^{2k+1}-1)}{(2k+2)!}B_{2k+2} \\ &= \sum_{2k<n}\binom{n}{2k}(-1)^k\frac{2^{2k+2}(2^{2k+1}-1)}{(2k+2)!}(-1)^k\frac{2(2k+2)!}{(2^{2k+2}-1)\pi^{2k+2}}\left(1+\frac{1}{3^{2k+2}}+\frac{1}{5^{2k+2}}+\cdots\right) \\ &= \sum_{2k<n}\binom{n}{2k}\frac{2^{2k+3}}{\pi^{2k+2}}\left(1+\frac{1}{3^{2k+2}}+\frac{1}{5^{2k+2}}+\cdots\right), \end{aligned}$$

so

$$A_n = \frac{8}{\pi^2}\sum_{2k<n}\sum_{j=0}^{\infty}\binom{n}{2k}\frac{2^{2k}}{(2j+1)^{2k+2}\pi^{2k}}. \tag{2.5}$$

Now consider the function

$$f_n(x) = \sum_{2k<n}\binom{n}{2k}x^{2k}. \tag{2.6}$$

It is easy to see that

$$f_n(x) = \frac{(1+x)^n+(1-x)^n-(1+(-1)^n)x^n}{2}. \tag{2.7}$$

Interchanging the order of summation in (2.5), we have

$$A_n = \frac{4}{\pi^2}\sum_{j=0}^{\infty}\frac{2}{(2j+1)^2}f_n\left(\frac{2}{(2j+1)\pi}\right),$$

or

$$A_n = \frac{4}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\left[\left(1+\frac{2}{(2j+1)\pi}\right)^n+\left(1-\frac{2}{(2j+1)\pi}\right)^n-(1+(-1)^n)\left(\frac{2}{(2j+1)\pi}\right)^n\right]. \quad (2.8)$$

For example,

$$A_1 = \frac{4}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\left[\left(1+\frac{2}{(2j+1)\pi}\right)+\left(1-\frac{2}{(2j+1)\pi}\right)\right]$$
$$= \frac{8}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)=\frac{8}{\pi^2}\frac{\pi^2}{8}=1,$$

and

$$A_2 = \frac{4}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\left[\left(1+\frac{2}{(2j+1)\pi}\right)^2+\left(1-\frac{2}{(2j+1)\pi}\right)^2-\left(\frac{2}{(2j+1)\pi}\right)^2\right]$$
$$= \frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}=1.$$

With formula (2.8) for A_n, the main term is where $j=0$. This gives

$$A_n \cong \frac{4}{\pi^2}\left[\left(1+\frac{2}{\pi}\right)^n+\left(1-\frac{2}{\pi}\right)^n-(1+(-1)^n)\left(\frac{2}{\pi}\right)^n\right]\cong\frac{4}{\pi^2}\left(1+\frac{2}{\pi}\right)^n.$$

For example, letting

$$C_n = \frac{4}{\pi^2}\left(1+\frac{2}{\pi}\right)^n,$$

consider the expanded table

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55
A_n	1	1	2	3	5	8	13.04	21.18	34.53	56.32
C_n	.66	1.09	1.8	2.91	4.76	7.79	12.75	20.86	34.14	55.88

Finally,

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n-\beta^n)\cong\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \cong .447(1.618)^n,$$

whereas

$$A_n \cong \frac{4}{\pi^2}\left(1+\frac{2}{\pi}\right)^n \cong .405(1.637)^n.$$

Thus, $A_n/F_n \cong .906(1.0115)^n$. Hence, for small n, $A_n \cong F_n$, although the A's grow exponentially faster than F_n in the long run.

3. THE BURGSTAHLER PROBLEM AS AN INVERSION PROBLEM

The real problem considered in this paper is the following: Find the sequence b_{2n+1} given that

$$F_n = \sum_{2k<n} \binom{n}{2k} b_{2k+1}.$$

This can be cast as a sum inversion problem: Given a known sequence $\{a_n\}_{n=0}^{\infty}$, suppose a new sequence is defined by

$$a_n = \sum_k c_{n,k} b_k,$$

for some given set of constants $c_{n,k}$, what can be said about the b's in terms of the a's? It must be pointed out that such a sequence of b's need not always exist. For example, if we attempt to define a sequence b_{2n+1} by

$$F_n = \sum_{2k\le n} \binom{n}{2k} b_{2k+1},$$

we find that there is no solution: $F_1 = b_1$, $F_2 = b_1 + b_3$, $F_3 = b_1 + 3b_3$ is an inconsistent system of three equations and two unknowns. Similarly, if we attempt to solve the system

$$n = \sum_{2k<n} \binom{n}{2k} b_{2k+1}$$

rather than the given one, we obtain $1 = b_1$, $2 = b_1$, and again there is no solution. In order to even ask Question 3 in the Introduction, we need

$$F_n = \sum_{2k<n} \binom{n}{2k} b_{2k+1}$$

to define a consistent system. In fact, as we will see, a proof that this system is consistent will give an affirmative answer to Question 2.

Here is a standard technique (see [2], Vol. I, pp. 437, 438, or [3], formula 2.1.2, p. 28) for solving a class of inversion problems: Suppose that

$$a_n = \sum_k c_{n,k} b_n,$$

where $c_{n,k}$ depends on only $n-k$, say $c_{n,k} = c_{n-k}$. In this case a_n is a convolution of b_n and c_n. Thus, passing to generating functions, with

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,\ B(x) = \sum_{n=0}^{\infty} b_n x^n,\ C(x) = \sum_{n=0}^{\infty} c_n x^n,$$

we have $A(x) = B(x)C(x)$. Hence, $B(x) = C(x)^{-1}A(x)$.

We use this technique to solve the inversion problem

$$a_n = \sum_{k=0}^{n-1} \binom{n}{k} b_k. \tag{3.1}$$

This expression only makes sense for $n \ge 1$; we extend it by setting $a_0 = 0$. Dividing each side by $n!$ gives

$$\frac{a_n}{n!} = \sum_{k=0}^{n-1} \frac{1}{(n-k)!} \frac{b_k}{k!}.$$

Here, the $n-1$ in the upper limit introduces a complication. The c_n in the convolution is

$$c_n = \begin{cases} 0, & n = 0, \\ \frac{1}{n!}, & n \geq 1. \end{cases}$$

In this case, $C(x) = e^x - 1$. Using exponential generating functions for a_n and b_n,

$$A(x) = B(x)(e^x - 1),$$

so

$$B(x) = \frac{1}{e^x - 1} A(x). \tag{3.2}$$

Since $A(0) = 0$, we can write this as

$$B(x) = \frac{1}{e^x - 1} \frac{1}{x} A(x).$$

Thus, the b's will be a convolution of Bernoulli numbers with the a's. In particular, we have

Theorem 3.1: Suppose that sequences $\{a_n\}$ and $\{b_n\}$ are defined by

$$a_n = \sum_{k=0}^{n-1} \binom{n}{k} b_k.$$

Then

$$b_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} B_{n-k} a_{k+1} = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k a_{n-k+1}.$$

We next consider the specific case where $a_n = F_n$. In this case, $A(x)$, the exponential generating function for the Fibonacci numbers is

$$A(x) = \frac{1}{\sqrt{5}}(e^{\alpha x} - e^{\beta x}).$$

Thus, we have

$$B(x) = \frac{1}{e^x - 1} \frac{1}{\sqrt{5}}(e^{\alpha x} - e^{\beta x}) = \frac{1}{\sqrt{5}} \frac{\sinh(\sqrt{5}x/2)}{\sinh x/2}. \tag{3.3}$$

Since $B(x)$ is an even function, all the odd terms are zero.

Theorem 3.2: If the sequence $\{c_n\}$ is defined by

$$F_n = \sum_{k=0}^{n-1} \binom{n}{k} c_k,$$

then $c_{2n+1} = 0$ for all n. Consequently,

$$F_n = \sum_{2k<n} \binom{n}{2k} c_{2k}. \tag{3.4}$$

Moreover,

$$F_n = \sum_{2k<n} \binom{n}{2k} x_k \tag{3.5}$$

has as its unique solution, $x_n = c_{2n}$ for all n.

Proof: The remarks preceding the theorem show that $c_{2n+1} = 0$ for all n, which gives us (3.4). As a consequence, we know that the system in (3.5) has a solution of the form $x_n = c_{2n}$. That this is the only solution follows by induction on n, using

$$F_{2n+1} = \sum_{2k<2n+1} \binom{2n+1}{2k} x_k = \sum_{k=0}^{n} \binom{2n+1}{2k} x_k,$$

or

$$x_n = \frac{1}{2n+1}\left(F_{2n+1} - \sum_{k=0}^{n-1} \binom{2n+1}{2k} x_k\right).$$

Corollary 3.3: The two systems of equations

$$F_{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{2k} x_k \tag{3.6}$$

and

$$F_{2n+2} = \sum_{k=0}^{n} \binom{2n+2}{2k} x_k \tag{3.7}$$

each have the same solution $x_n = c_{2n}$ for all n.

Proof: Again, a solution $x_n = c_{2n}$ exists to each of these systems and, by induction, each has a unique solution.

Dr. Burgstahler's numbers b_{2n+1} are now just c_{2n} above. Combining Theorems 3.1 and 3.2, we have

Theorem 3.4: The system of equations

$$F_n = \sum_{2k<n} \binom{n}{2k} b_{2k+1}$$

is consistent and has a unique solution

$$b_{2n+1} = \frac{1}{2n+1} \sum_{k=0}^{2n} \binom{2n+1}{k} B_k F_{2n-k+1}.$$

We are now in a position to answer Dr. Burgstahler's second question: as coefficients in $\tan x$ are changed one by one to the b_{2n+1}, each change corrects two terms to Fibonacci numbers. This is because of Corollary 3.3, which indicates that both F_{2n+1} and F_{2n+2} can be expressed as sums involving $b_1, b_3, \ldots, b_{2n+1}$.

4. CONCLUDING REMARKS

We have not yet given a complete answer to Question 3. While we have given a formula for the terms of the sequence $\{b_{2n+1}\}$, we have not said anything about the function $f(x) = b_1 x + b_3 x^2 + b_5 x^5 + \cdots$.

Theorem 4.1: The power series $\sum_{n=0}^{\infty} b_{2n+1}x^{2n+1}$ has a radius of convergence of 0.

Sketch of Proof: Suppose, by way of contradiction, that this is not the case. That is, suppose that $\sum_{n=0}^{\infty} b_{2n+1}x^{2n+1}$ converges to a function $f(x)$, at least for $|x|<C$ for some constant $C>0$. Then it may be shown that $f(x)$ satisfies the functional equation

$$f(x)=\frac{x^2}{1+x-x^2}+f\left(\frac{x}{1+x}\right), \tag{4.1}$$

in some neighborhood of the origin. Since $1+x-x^2=0$ at $x=\alpha$, $x=\beta$, this region must be a subset of the interval (β,α). However, given a function f satisfying (4.1), if $x=a$ is a pole for f, then so is $\frac{x}{1-x}=a$ or $x=\frac{a}{1-a}$. Iterating this, f has a pole at each of the values $x=\frac{a}{1-ka}$, if it has a pole at a. In particular, for $a=\beta$, this gives an increasing sequence of poles with 0 as its limit. As no convergent power series about the origin can have this property, we have a contradiction.

Thus, the first part of Question 3 was slightly naive—there was no guarantee that such a function $f(x)$ even existed; in fact, one does not. However, it was only by following the generating function approach above, and noting the problem of the poles that the author discovered this fact.

One may ask about an exponential generating function for the sequence $\{b_{2n+1}\}$ rather than the ordinary generating function, of course. As a consequence of formula (3.3), this exponential generating function is

$$\int_0^x \frac{t(e^{\alpha t}-e^{\beta t})}{\sqrt{5}(e^t-1)}dt \quad \text{or} \quad \int_0^x \frac{t\sinh(\sqrt{5}t/2)}{\sqrt{5}\sinh t/2}dt.$$

The integral is needed to correct the index from c_{2n} to b_{2n+1}.

It is reasonable to ask when the system of equations

$$a_n=\sum_{2k<n}\binom{n}{2k}x_k \tag{4.2}$$

is consistent. We have the following result.

Theorem 4.2: The system in (4.2) is consistent if and only if the solution to the system

$$a_n=\sum_{k=0}^{n-1}\binom{n}{k}y_k \tag{4.3}$$

satisfies the condition $y_{2n+1}=0$ for all n. In this case, the solution to (4.2) is given by $x_n=y_{2n}$ for all n.

Proof: If the solution to (4.3) satisfies the condition that $y_{2n+1}=0$ for all n, then we obtain existence and uniqueness for solutions to (4.2) in exactly the same way as in Theorem 3.2. For the other direction, we assume that (4.2) has a solution and proceed in induction on n to show that in the solution to (4.3) all y_{2n+1} are 0 and that $y_{2n}=x_n$ for all n. To begin the induction, the equations $a_1=x_0$ and $a_2=x_0$ show that to be consistent, we need $a_1=a_2$. In this case, $y_0=a_1$, $y_0+2y_1=a_2$ gives $y_1=0$. Moreover, since $x_0=a_1$, we have that $x_0=y_0$.

So, by way of induction, assume that, for $0\le k\le n-1$, $y_{2k+1}=0$ and $x_k=y_{2k}$. We have

$$a_{2n+1} = \sum_{2k<2n+1} \binom{2n+1}{2k} x_k = \sum_{k=0}^{n} \binom{2n+1}{2k} x_k,$$

and

$$a_{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{2k} y_k = \sum_{k=0}^{n} \binom{2n+1}{2k} y_{2k}.$$

Since $y_{2k} = x_k$ for all $k < n$, comparing these two expressions gives that $y_{2n} = x_n$. Now

$$a_{2n+2} = \sum_{2k<2n+2} \binom{2n+2}{2k} x_k = \sum_{k=0}^{n} \binom{2n+2}{2k} x_k$$

and

$$a_{2n+2} = \sum_{k=0}^{2n+1} \binom{2n+2}{k} y_k = \sum_{k=0}^{n} \binom{2n+2}{2k} y_{2k} + (2n+2)y_{2n+1}$$

force y_{2n+1} to be 0. This completes the proof.

We may now use generating function techniques to give more information.

Corollary 4.3: The system in (4.2) is consistent if and only if the exponential generating function $A(x)$ for $\{a_n\}$ satisfies the functional equation

$$A(x) = -e^x A(-x). \tag{4.4}$$

Proof: We may solve system (3.1) rather than (4.2). By formula (3.2), we have the relation

$$B(x) = \frac{1}{e^x - 1} A(x),$$

where $B(x)$ is the exponential generating function for the y_n. By the previous theorem, $B(x)$ must be an even function of x. Hence,

$$\frac{1}{e^{-x} - 1} A(-x) = \frac{1}{e^x - 1} A(x),$$

from which the functional equation follows.

The functional equation (4.4) does not place too heavy a restriction on sequences $\{a_n\}$. For example, if $f(x)$ is any odd function, then $\frac{2e^x}{e^x+1} f(x)$ will satisfy equation (4.4). We conclude with the following result.

Theorem 4.4: If the sequence $\{a_n\}$ satisfies a recurrence relation of the type $a_n = a_{n-1} + ca_{n-2}$, where c is an arbitrary constant and $a_0 = 0$, then system (4.2) is consistent.

Proof: The case where $c = 0$ is trivial; the solution to the recurrence relation being just the 0 sequence. Another special case is $c = \frac{-1}{4}$, in which case one may check that

$$a_n = n2^{-n}, \quad A(x) = \frac{x}{2} e^{x/2}, \quad \text{and} \quad B(x) = \frac{x}{2} \frac{e^{x/2}}{e^x - 1}.$$

In the cases where $c \neq 0$, $\frac{-1}{4}$, any solution satisfying $a_0 = 0$ will be of the form $a_n = C(u^n - v^n)$, where C is a constant, and $u + v = 1$ (u and v being the solutions to $x^2 - x - c = 0$). In this case, $A(x) = C(e^{ux} - e^{vx})$, so

$$-e^x A(-x) = -Ce^x(e^{-ux} - e^{-vx}) = C(e^{(1-v)x} - e^{1-u)x}) = C(e^{ux} - e^{vx}) = A(x),$$

so $A(x)$ satisfies the required functional equation, completing the proof.

As a very easy example, if $c = 2$, one may check that $a_n = 2^n - (-1)^n$ produces a consistent system for (4.2). In this case,

$$b_0 = 3 \quad \text{and} \quad b_n = \begin{cases} 0, & n \text{ odd}, \\ 2, & n \text{ even}, n > 0. \end{cases}$$

That this works can be independently checked using formulas (2.6) and (2.7).

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