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A Character Sum Evaluation and Gaussian Hypergeometric Series

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Evans has conjectured the value of a certain character sum. The conjecture is confirmed using properties of Gaussian hypergeometric series which are well known for hypergeometric series. Several related questions are discussed. © 1986 Academic Press, Inc.

1. INTRODUCTION

Given the finite field $GF(p)$, let ϕ be the usual quadratic residue character

$$\phi(x) = \begin{cases} 1, & 0 \neq x \in GF(p) \quad \text{is a square,} \\ -1, & 0 \neq x \in GF(p) \quad \text{is not a square} \\ 0, & x = 0. \end{cases} \tag{1.1}$$

In [6, Eq. (20)] the following character sum evaluation was conjectured

$$\sum_{x, y \in GF(p)} \phi(1+x)\,\phi(1+y)\,\phi(x)\,\phi(y)\,\phi(x+y) = \begin{cases} -p\phi(2), & p \equiv 5 \text{ or } 7 \pmod 8, \\ \phi(2)(4c^2-p), & p \equiv 1 \text{ or } 3 \pmod 8, \end{cases} \tag{1.2}$$

where $p = c^2 + 2d^2$ uniquely for $p \equiv 1$ or $3 \pmod 8$. In this paper we verify (1.2) and generalize (1.2) for $p \equiv 1 \pmod 4$.

Our basic idea is to recognize the left-hand side of (1.2) as a Gaussian analogue of a generalized hypergeometric series. It is well known [1]

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which generalized hypergeometric series are evaluable. The Gaussian analogues for evaluations and transformations of ${}_2F_1$'s and ${}_3F_2$'s are given in [8]. Thus we consider (1.2) as an exercise in generalized hypergeometric series. We believe this is a powerful technique for character sum evaluations.

First, we recall some notation and conventions. Given a multiplicative character A of $GF(p)$, we extend A to all of $GF(p)$ by defining $A(0)=0$. Let δ be the function on $GF(p)$ whose only non-zero value is $\delta(0)=1$. Then the multiplicative characters of $GF(p)$ and δ form a basis for complex-valued functions on $GF(p)$. Let ε be the trivial multiplicative character, $\varepsilon(x)=1$, $x\neq 0$.

For multiplicative characters A and B let $G(A)$ [9, p. 90] denote the Gauss sum of A and $J[A, B]$ denote the Jacobi sum of A and B. We shall frequently use the basic facts about these sums: the reflection formula [9, p. 92]

$$G(A)\,G(\bar{A})=pA(-1), \qquad A\neq\varepsilon, \tag{1.3}$$

the Jacobi sum evaluation [9, p. 93, Theorem 1],

$$J[A, B]=\frac{G(A)\,G(B)}{G(AB)}, \qquad AB\neq\varepsilon \tag{1.4}$$

and the Hasse–Davenport formula [12, p. 477, Eq. (3)],

$$G(A^n)\,G(\varphi)\cdots G(\varphi^{n-1})=A^n(n)\,G(A)\cdots G(A\varphi^{n-1}), \tag{1.5}$$

where φ is a multiplicative character of $GF(p)$ of order n.

In Section 2 we define Gaussian hypergeometric series and give the "integral representation" for such series which are relevant to (1.2). By applying a ${}_3F_2$ evaluation, we verify (1.2) for $p\equiv 1 \pmod 4$ in Section 3. In that section, a simple proof of (1.2) for $p\equiv 5$ or $7 \pmod 8$ from a ${}_3F_2$ transformation is also given. For all values of p the conjecture is verified in Section 4. For $p\equiv 1 \pmod 4$ (1.2) can be generalized. We state such a result in (5.3).

2. The Gaussian Analogue of a ${}_3F_2$

Recall that a ${}_3F_2$ generalized hypergeometric series is defined by

$${}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}\,\middle|\, x\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(c)_k}{k!(d)_k(e)_k}x^k, \tag{2.1}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$. This can be rewritten with binomial coefficients as

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix} \,\middle|\, x\right) = \alpha \sum_{k=0}^{\infty} \binom{a+k-1}{k} \binom{b+k-1}{d+k-1} \binom{c+k-1}{e+k-1} x^k, \quad (2.2) $$

where α is a constant independent of x. We shall see that the left-hand side of (1.2) is a Gaussian analogue of (2.2). First we need the Gaussian analogue of the binomial theorem.

PROPOSITION 2.3 (Gaussian binomial theorem). *Let A be a multiplicative character of $GF(p)$. For $x \in GF(p)$,*

$$ A(1+x) = \delta(x) + \frac{p}{p-1} \sum_{\chi} \binom{A}{\chi} \chi(x), $$

where the sum is over all multiplicative characters χ of $GF(p)$, and

$$ \binom{A}{\chi} = \frac{1}{p} J[A, \bar{\chi}]\, \chi(-1). $$

Proof. Since $A(1+x)$ is a complex function on $GF(p)$, clearly there is an expansion

$$ A(1+x) = \delta(x) + \sum_{\chi} c(\chi, A)\, \chi(x) $$

for some complex numbers $c(\chi, A)$. By orthogonality,

$$ (p-1)\, c(\chi, A) = \sum_{x \in GF(p)} A(1+x)\, \bar{\chi}(x) = \chi(-1)\, J[A, \bar{\chi}], $$

so

$$ c(\chi, A) = \frac{p}{p-1} \binom{A}{\chi}. \quad ∎ $$

It is clear that Proposition 2.3 also holds for $GF(q)$, where $q = p^n$. Moreover, if χ and $A\bar{\chi} \neq \varepsilon$, we have

$$ \binom{A}{\chi} = \frac{G(A)\, G(\bar{\chi})\, \chi(-1)}{G(A\bar{\chi})\, p} = \frac{G(A)}{G(A\bar{\chi})\, G(\chi)} \quad (2.4) $$

which corresponds to

$$ \binom{n}{k} = \frac{n!}{k!(n-k)!}. \quad (2.5) $$

More information on the analogy can be found in [8, Chap. 2].

Three simple formulas we will find useful are

$$\binom{A}{B} = \binom{A}{A\bar{B}}, \tag{2.6}$$

$$\binom{A}{B} = \binom{B\bar{A}}{B} B(-1), \tag{2.7}$$

and

$$\binom{A}{B} = \binom{\bar{B}}{\bar{A}} AB(-1). \tag{2.8}$$

Definition 2.9. For multiplicative characters A, B, C, D, and E of $GF(p)$ and $x \in GF(p)$, let

$${}_3F_2\left(\begin{matrix} A, B, C \\ D, E \end{matrix} \,\middle|\, x\right) = \frac{p}{p-1} \sum_{\chi} \binom{A\chi}{\chi}\binom{B\chi}{D\chi}\binom{C\chi}{E\chi} \chi(x).$$

There are two minor differences with (2.2). The constant α has been replaced with $p/(p-1)$. The index set k in (2.2) has been replaced by multiplicative characters χ. These characters form a finite cyclic group, thus can be considered as points on a circle. Since $\chi(0) = 0$ for all χ; in Definition 2.9, ${}_3F_2(0) = 0$, while in (2.1), ${}_3F_2(0) = 1$. One might ask why (2.1) is not used to define a Gaussian ${}_3F_2$ as a sum of quotients of Gauss sums. These two definitions of Gaussian ${}_3F_2$'s do not agree because of the exceptional cases in the Jacobi sum evaluation. Definition 2.9 leads to more compact results because the Gaussian binomial theorem (Proposition 2.3) is easier to state with Jacobi sums than with Gauss sums.

Next we need a result which is an analogue of a double integral representation of a terminating ${}_3F_2$.

Proposition 2.10. *For multiplicative characters A, B, C, D, and E of $GF(p)$ and $0 \neq t \in GF(p)$,*

$$\begin{aligned} I &= \sum_{x, y \in GF(p)} A(1+x)\, B(1+y)\, C(x+ty)\, D(x)\, E(y) \\ &= p^2 BCDE(-1)\, {}_3F_2\left(\begin{matrix} \bar{C}, \overline{ACD}, E \\ \overline{CD}, BE \end{matrix} \,\middle|\, -t\right). \end{aligned}$$

Proof. Because $D(0) = 0 = E(0)$, we can assume $x \neq 0 \neq y$. Expand

$A(1+x)$, $B(1+y)$, and $C(x+ty)=C(x)\,C(1+ty/x)$ by the binomial theorem to obtain

$$I=\left(\frac{p}{p-1}\right)^3 \sum_{\alpha,\beta,\gamma}\sum_{x,y}\binom{A}{\alpha}\binom{B}{\beta}\binom{C}{\gamma}(CD\alpha)(x)(E\beta)(y)\,\gamma(ty/x). \qquad (2.11)$$

By orthogonality the x and y sums are zero unless $\bar{\gamma}CD\alpha=\varepsilon$ and $\gamma E\beta=\varepsilon$, so

$$I=\frac{p^3}{p-1}\sum_{\gamma}\binom{C}{\gamma}\binom{A}{\overline{CD}\gamma}\binom{N}{\overline{E}\gamma}\gamma(t)$$

The result follows from (2.7) and (2.8).

3. The Conjecture as a $_3F_2$ Evaluation

Certain generalized hypergeometric functions can be evaluated as quotients of gamma functions [1]. The Gaussian analogous of these evaluations should have quotients of Gauss sums, or products of Gaussian binomial coefficients. From Proposition 2.10, any Gaussian $_3F_2$ evaluation gives a character sum evaluation. The conjecture (1.2) will be a special case in which the Gaussian binomial coefficients themselves are evaluable.

There are four major $_3F_2$ evaluations [4, Sect. 4.4]. All of them have Gaussian analogues [8, p. 126]:

Saalschütz's theorem,

$${}_3F_2\left(\begin{matrix}A, B, C\\ D, ABC\overline{D}\end{matrix}\,\middle|\,1\right)$$

Dixon's theorem,

$${}_3F_2\left(\begin{matrix}A, B, C\\ A\overline{B}, A\overline{C}\end{matrix}\,\middle|\,1\right)$$

Watson's theorem,

$${}_3F_2\left(\begin{matrix}A^2, B^2, C\\ \phi AB, C^2\end{matrix}\,\middle|\,1\right),$$

and Whipple's theorem,

$${}_3F_2\left(\begin{matrix}A, \overline{A}, B\\ C, B^2\overline{C}\end{matrix}\,\middle|\,1\right).$$

Proposition 2.10 implies that (1.2) is equivalent to an evaluation of

$$ {}_3F_2\left(\begin{matrix}\phi, \phi, \phi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, -1\right), $$

which is none of the above four ${}_3F_2$'s.

However, Whipple has also evaluated [1, p. 97, Ex. 3(i)]

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}+x, \frac{1}{2}-x \\ 1-x, 1+x\end{matrix} \,\middle|\, -1\right) $$
$$ = \pi^2 x/\sqrt{2} \sin \pi x \Gamma(\tfrac{1}{2}x+\tfrac{5}{8})\, \Gamma(\tfrac{1}{2}x+\tfrac{7}{8})\, \Gamma(\tfrac{5}{8}-\tfrac{1}{2}x)\, \Gamma(\tfrac{7}{8}-\tfrac{1}{2}x). \quad (3.1) $$

We need the Gaussian analogue of this evaluation for $x=0$. Note the eights in (3.1) and in (1.2).

One way to prove (3.1) is to put $a=\frac{1}{2}$, $b=\frac{1}{2}+x$, $c=\frac{1}{2}-x$, and $x=-1$ in the quadratic ${}_3F_2$ transformation [1, p. 97, Ex. 4(iv)] to obtain

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}+x, \frac{1}{2}-x \\ 1-x, 1+x\end{matrix} \,\middle|\, -1\right) = 2^{-1/2}\, {}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{3}{4}, \frac{1}{2} \\ 1-x, 1+x\end{matrix} \,\middle|\, 1\right). \quad (3.2) $$

The right-hand side of (3.2) is evaluable by Whipple's theorem [1, p. 16] for a ${}_3F_2(1)$. We concentrate on the Gaussian analogue of this proof for the rest of the section. It will prove (1.2) if $p \equiv 1 \pmod 4$.

The Gaussian analogue of the ${}_3F_2$ quadratic transformation is known [8, p. 70, Theorem 5.23]. We do not state it here, because the general case involves several extra terms for special values of the parameters. The Gaussian analogue of (3.2) for $x=0$ that results is

$$ {}_3F_2\left(\begin{matrix}\phi, \phi, \phi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, -1\right) = \frac{p\phi(2)}{p-1} \sum_\theta \binom{\phi\theta^2}{\theta}\binom{\phi\theta}{\theta}\binom{\phi\theta}{\theta} \bar{\theta}(4) $$
$$ + \frac{\phi(-1)}{p} - \frac{\phi(2)}{p^2}. \quad (3.3) $$

The θ-sum is the Gaussian analogue of right-hand side of (3.2) for $x=0$. In fact, if $\phi=\alpha^2$ is a square ($p \equiv 1$ or $5 \pmod 8$), the Hasse–Davenport formula implies that

$$ \binom{\phi\theta^2}{\theta} = \binom{\alpha^2\theta^2}{\theta} = (\alpha\theta)(4) \binom{\alpha\theta}{\bar{\alpha}}\binom{\alpha\phi\theta}{\theta} \Big/ \binom{\phi}{\bar{\alpha}} \quad (3.4) $$

and the θ-sum is a ${}_3F_2$. In this case $\phi(2)=\alpha(-1)$ and $\phi(-1)=1$ so that (3.3) becomes

$$ {}_3F_2\left(\begin{matrix}\phi, \phi, \phi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, -1\right) = \alpha(-1)\, {}_3F_2\left(\begin{matrix}\alpha\phi, \alpha, \phi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, 1\right) + \frac{1}{p}. \quad (3.5) $$

It remains to evaluate the Gaussian ${}_3F_2(1)$ by Whipple's theorem. The special case that we need is [8, p. 83, Theorem 5.45(ii)],

$$
{}_3F_2\left(\begin{matrix}\alpha\phi, \alpha, \phi\\ \varepsilon, \varepsilon\end{matrix}\,\middle|\, 1\right)
= \begin{cases} 0, & \alpha\phi \text{ is not a square } (p \equiv 5 \bmod 8) \\ \binom{\phi}{\theta}\binom{\phi}{\alpha\phi\theta} + \binom{\phi}{\phi\theta}\binom{\phi}{\alpha\theta}, & \alpha\phi = \theta^2 \ (p \equiv 1 \bmod 8). \end{cases} \tag{3.6}
$$

So, if $p \equiv 5 \pmod 8$, (3.5), (3.6) and Proposition 2.10 imply that $I = p$ and the conjecture is verified. For $p \equiv 1 \pmod 8$, we see that

$$
I = p + p^2\alpha(-1)\left[\binom{\phi}{\theta}\binom{\phi}{\alpha\phi\theta} + \binom{\phi}{\phi\theta}\binom{\phi}{\alpha\theta}\right]. \tag{3.7}
$$

Clearly $\theta = \bar{\alpha}\bar{\theta}$ and (2.6) imply

$$
\binom{\phi}{\alpha\phi\theta} = \binom{\phi}{\theta} \quad \text{and} \quad \binom{\phi}{\alpha\theta} = \binom{\phi}{\phi\theta} = \binom{\phi}{\bar{\theta}},
$$

so

$$
I = p + p^2\alpha(-1)\left[\binom{\phi}{\theta}^2 + \binom{\phi}{\bar{\theta}}^2\right], \tag{3.8}
$$

which, by Proposition 2.3, reduces to

$$
I = p + J[\phi, \bar{\theta}]^2 + J[\phi, \theta]^2. \tag{3.9}
$$

It is easy to show that $J[\phi, \theta] = c + di\sqrt{2}$, $c, d \in \mathbb{Z}$. (In fact, $|J|^2 = p$ proves $p = c^2 + 2cd^2$.) Since $\bar{\phi} = \phi$, $J[\phi, \bar{\theta}] = c - di\sqrt{2}$. Thus (3.9) is

$$
I = p + (c - id\sqrt{2})^2 + (c + di\sqrt{2})^2 = 4c^2 - p. \tag{3.10}
$$

This verifies (1.2) for $p \equiv 1 \pmod 8$.

There is an easy proof of (1.2) for $p \equiv 5$ or $7 \pmod 8$. In this case we shall show that

$$
\frac{p}{p-1}\sum_{\theta}\binom{\phi\theta^2}{\theta}\binom{\phi\theta}{\theta}\binom{\phi\theta}{\theta}\bar{\theta}(4) = 1/p^2. \tag{3.11}
$$

Then (3.11), (3.3), Proposition 2.10, and $\phi(-2) = -1$ show that (1.2) holds.

The idea of our proof of (3.11) is to find an analogue of a ${}_3F_2$ transformation which maps the θ-sum to itself. Recall that the θ-sum corresponds to

$${}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{3}{4}, \frac{1}{2}\\ 1, 1\end{matrix}\,\middle|\, 1\right),$$

and that all Gaussian hypergeometric series correspond to terminating hypergeometric series. So if $f(x)$ is the Gaussian analogue of

$${}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{3}{4}\\ 1\end{matrix}\,\middle|\, x\right)$$

Kummer's linear transformation [1, p. 4, Eq. (1)], should imply that "$f(x) = f(1-x)$," up to correction terms. In fact, let

$$f(x) = \frac{p}{p-1}\sum_{\chi}\binom{\phi\chi^2}{\chi}\binom{\phi\chi}{\chi}\chi\left(\frac{x}{4}\right). \tag{3.12}$$

The Gaussian analogue of Kummer's linear transformation is ($A = \phi$, $B = \varepsilon$ in [8, p. 109, (7.7)])

$$\begin{aligned} f(x) = \phi(-2) f(1-x) &- \delta(1-x)\,\phi(-2)/p \\ &- \phi(-x)/p + \phi(2)\,\phi(1-x)/p + \delta(x)/p. \end{aligned} \tag{3.13}$$

If we multiply (3.13) by $\phi(x)\,\phi(1-x)$, sum on x, use the definition of the binomial coefficient in Proposition 2.3, and rearrange the θ-sums, we have

$$(1-\phi(-2))\frac{p}{p-1}\sum_{\theta}\binom{\phi\theta^2}{\theta}\binom{\phi\theta}{\theta}\binom{\phi\theta}{\theta}\bar{\theta}(4) = (1-\phi(-2))/p^2. \tag{3.14}$$

Thus, if $\phi(-2) = -1$, i.e., $p \equiv 5$ or $7 \pmod 8$, we have (3.11).

E. Lehmer also verified the $p \equiv 5$ or $7 \pmod 8$ cases by a change of variables in (1.2). Because (3.13) follows from a change of variables in the "integral representation" for $f(x)$, our proof is essentially the same.

In the next section we give a proof of (1.2) for all p. The function $f(x)$ defined by (3.12) will be important.

4. A Proof of the Conjecture

In Section 3 we used Whipple's ${}_3F_2(1)$ evaluation to verify the conjecture for $p \equiv 1$ or $5 \pmod 8$. Our proof did not work for $p \equiv 3$ or $7 \pmod 8$

because we could not use the Hasse–Davenport formula to reduce the right-hand side of (3.3) to a Gaussian ${}_3F_2(1)$. In this section we give another proof of (3.1) for $x=0$ whose Gaussian analogue does not depend upon p.

A theorem of Clausen is [1, p. 86, (4)],

$$ {}_3F_2\left(\begin{matrix}2\alpha, 2\beta, \alpha+\beta\\ 2\alpha+2\beta, \alpha+\beta+\frac{1}{2}\end{matrix}\,\middle|\, x\right) = {}_2F_1\left(\begin{matrix}\alpha, \beta\\ \alpha+\beta+\frac{1}{2}\end{matrix}\,\middle|\, x\right)^2. \tag{4.1} $$

If $\alpha=\beta=\frac{1}{4}$ and $x=-1$, the ${}_2F_1$ is evaluable by Kummer's theorem [1, p. 9] to give (3.1) for $x=0$. For Gaussian analogues, this proof will work as long as the analogue of $\frac{1}{4}$, α, exists. So again it appears that we need $p\equiv 1$ or $5 \pmod 8$. However, by applying a linear ${}_2F_1$ transformation [1, p. 10, Eq. (2.4, (1)] our special case of (4.1) becomes

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ 1, 1\end{matrix}\,\middle|\, x\right) = (1-x)^{-1/2}\, {}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{3}{4}\\ 1\end{matrix}\,\middle|\, \frac{x}{x-1}\right)^2. \tag{4.2} $$

Clearly

$$ {}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{3}{4}\\ 1\end{matrix}\,\middle|\, \frac{x}{x-1}\right) = \sum_{k=0}^{\infty} \frac{(1/2)_{2k}}{k!\,k!}\left(\frac{x}{4(x-1)}\right)^k, \tag{4.3} $$

so that a Gaussian analogue of this ${}_2F_1$ can be given without referring to α

$$ \text{“}{}_2F_1\left(\begin{matrix}\alpha, \alpha\phi\\ \varepsilon\end{matrix}\,\middle|\, \frac{x}{x-1}\right)\text{”} = \frac{p}{p-1}\sum_{\chi}\binom{\phi\chi^2}{\chi}\binom{\phi\chi}{\chi}\chi\left(\frac{x}{4(x-1)}\right) = f\left(\frac{x}{x-1}\right). \tag{4.4} $$

So there should be a Gaussian analogue of (4.2) with (4.4) replacing the ${}_2F_1$. Moreover, (4.2) looks attractive because of the square which also occurs in (1.2) in c^2.

The appropriate Gaussian analogue of (4.2) with $u=x/(x-1)$ is [8, p. 94, Proposition 6.14],

$$ \phi((1-u)/u)\, {}_3F_2\left(\begin{matrix}\phi, \phi, \phi\\ \varepsilon, \varepsilon\end{matrix}\,\middle|\, \frac{u}{u-1}\right) $$

$$ = \phi(u) f(u)^2 + 2\frac{\phi(-1)}{p} f(u) - \frac{p-1}{p^2}\phi(u) + \frac{p-1}{p^2}\delta(1-u). \tag{4.5} $$

If we put $u=\frac{1}{2}$ and

$$ \frac{p}{p-1}\sum_{\chi}\binom{\phi\chi^2}{\chi}\binom{\phi\chi}{\chi}\bar{\chi}(8) = A = f(\tfrac{1}{2}) \tag{4.6} $$

then (4.5) becomes

$$ {}_3F_2\left(\begin{matrix}\phi, \phi, \phi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, -1\right) = \phi(2)\,A^2 + 2\phi(-1)\,Ap^{-1} - \phi(2)(p-1)\,p^{-2}, \quad (4.7) $$

and we need only evaluate A in (4.6). It is an analogue of a ${}_2F_1(1/2)$ evaluation. The next proposition gives the analogue of the integral representation for A.

PROPOSITION 4.8. *If A is defined by* (4.6), *then*

$$ A = \frac{1}{p} \sum_{y \in GF(p)} \phi(1-y)\,\phi(1-2y^2) - \frac{\phi(-2)}{p}. $$

Proof. In (4.6) use the Jacobi sum definition of $\binom{\phi\chi^2}{\chi} = \binom{\phi\bar{\chi}}{\chi}\chi(-1)$ to find

$$ A = \frac{1}{p-1} \sum_{\chi} \binom{\phi\chi}{\chi} \bar{\chi}(8) \sum_{x \in GF(p)} \bar{\chi}(1-x)(\phi\bar{\chi})(x), $$

or

$$ A = \frac{1}{p-1} \sum_{x \in GF(p)} \phi(x) \sum_{\chi} \binom{\phi\chi}{\chi} \bar{\chi}(8x(1-x)). \quad (4.9) $$

Clearly (2.7) and the binomial theorem imply

$$ \frac{p}{p-1} \sum_{\chi} \binom{\phi\chi}{\chi} \bar{\chi}(8x(1-x)) = \phi\left(1 - \frac{1}{8x(1-x)}\right), \qquad x \neq 0, 1. \quad (4.10) $$

So

$$ A = \frac{1}{p} \sum_{x \in GF(p)} \phi(x)\,\phi(8x(1-x))\,\phi(8x(1-x)-1). \quad (4.11) $$

If $x = (1+y)/2$, since $\phi^2(x) = 1$, $x \neq 0$, we have

$$ A = \frac{1}{p} \sum_{\substack{y \in GF(p) \\ y \neq -1}} \phi(1-y)\,\phi(2(1-y^2)-1). \quad (4.12) $$

The $y = -1$ term contributes $\phi(-2)/p$ to A. ∎

PROPOSITION 4.13.

$$\sum_{y \in GF(p)} \phi(1-y)\,\phi(1-2y^2) = \begin{cases} 0, & p \equiv 5 \textit{ or } 7 \pmod 8) \\ \pm 2c, & p \equiv 1 \textit{ or } 3 \pmod 8), \end{cases}$$

where $p = c^2 + 2d^2$.

Proof. Clearly

$$\sum_{y \in GF(p)} \phi(1-2y^2) = 1 + \sum_{w \neq 0} (1+\phi(w))\,\phi(1-2w)$$
$$= \sum_{w \in GF(p)} \phi(w)\,\phi(1-2w), \tag{4.14}$$

which implies

$$\sum_{y \in GF(p)} \phi(1-2y^2) = \phi(2)\,J[\phi, \phi] = -\phi(-2). \tag{4.15}$$

Also, similarly,

$$\sum_{y \in GF(p)} [1+\phi(1-y)]\,\phi(1-2y^2) = \sum_{w \in GF(p)} \phi(1-2(1-w^2)^2). \tag{4.16}$$

Since $\phi(1-2(1-w^2)^2) = \phi(-2+4w^2-w^2)$, $w \neq 0$,

$$\sum_{y \in GF(p)} [1+\phi(1-y)]\,\phi(1-2y^2)$$
$$= \phi(-1) - \phi(-2) + \phi(-1) \sum_{w \in GF(p)} \phi(w^4 - 4w^2 + 2). \tag{4.17}$$

The Brewer polynomial $V_4(w, 1)$ [3, Sect. 5] is precisely $w^4 - 4w^2 + 2$, so [3, Sect. 5.2], (4.17), and (4.15) imply

$$\sum_{y \in GF(p)} \phi(1-y)\,\phi(1-2y^2) = \phi(-1) + \phi(-1)\,\Lambda_4. \tag{4.18}$$

The value of Λ_4, in [3, Theorem 5.17] completes the proof. ∎

To complete the proof of (1.2), use Propositions 4.13, 4.8, and 2.10 and (4.7).

5. Concluding Remarks

There are other Gaussian ${}_3F_2(-1)$ evaluations [8, Chap. 6]. For example, if $p \equiv 1 \pmod 4$, then a Gaussian analogue of (3.1) is

$$
{}_3F_2\left(\begin{matrix}\phi\bar{A},\ \phi A,\ \phi \\ A,\ \bar{A}\end{matrix}\ \middle|\ -1\right)
= \begin{cases} \dfrac{1}{p} & \text{if } \alpha\phi A \text{ is not square,} \\ (AD)(-1)\dbinom{\phi D}{\phi\alpha}\dbinom{\phi\alpha}{D} + AD(-1)\dbinom{D}{\phi}\dbinom{\phi}{\alpha D} + \dfrac{1}{p} & \text{if } \alpha\phi A = D^2. \end{cases} \tag{5.1}
$$

Equation (5.1) follows from Whipple's ${}_3F_2(1)$ evaluation as in Section 3. We have assumed in (5.1) that $p \equiv 1 \pmod 4$ and $A \neq \alpha$, ϕ, or $\alpha\phi$. The binomial coefficients can be reduced to Gauss sums

$$
\binom{\phi D}{\phi\alpha}\binom{\phi\alpha}{D} + \binom{D}{\phi}\binom{\phi}{\phi D} = \frac{pD(-1)}{G(D)\,G(\phi\bar{D})\,G(\alpha\phi D)\,G(\alpha\phi\bar{D})} + \frac{pD(-1)}{G(\phi D)\,G(\bar{D})\,G(\alpha D)\,G(\alpha\bar{D})}, \tag{5.2}
$$

so that if $\alpha\phi A = D^2$, (5.1) becomes $(\alpha(-1) = \phi(2) = A(-1))$,

$$
{}_3F_2\left(\begin{matrix}\phi\bar{A},\ \phi A,\ \phi \\ A,\ \bar{A}\end{matrix}\ \middle|\ -1\right) = \frac{1}{p} + \frac{p\phi(2)}{G(D)\,G(\phi\bar{D})\,G(\alpha\phi D)\,G(\alpha\phi\bar{D})} + \frac{p\phi(2)}{G(\phi D)\,G(\bar{D})\,G(\alpha D)\,G(\alpha\bar{D})}. \tag{5.3}
$$

This is the analogue of (3.1) since D corresponds to $3/8 + x/2$, p corresponds to π, and $\phi(2)$ corresponds to $\sqrt{2}$. These Gaussian binomial coefficients are not evaluable, for general A. In fact, many of the results of [2] and [3] can be interpreted as evaluating special Gaussian binomial coefficients.

Any ${}_3F_2$ evaluation is a character sum evaluation. Dixon's theorem for a ${}_3F_2$ is equivalent to the Gaussian analogue of the two-dimensional Selberg integral [5]. We mention the Gaussian analogue of a "strange" ${}_3F_2$ evaluation [7, Eq. (1.1)]

$$
{}_3F_2\left(\begin{matrix} A^2, A^6\bar{B}^3, \bar{B}^3 \\ \phi A^3, A^3 \end{matrix} \middle| \frac{3}{4}\right)
= A^3(-4)\binom{A^3B^3}{\phi}\begin{cases} \binom{A^2B}{B}, & p \not\equiv 1 \pmod 3 \\ \binom{A^2B}{B} + \binom{A^2B\psi}{B\psi} + \binom{A^2B\psi^2}{B\psi^2}, & p \equiv 1 \pmod 3, \end{cases} \tag{5.4}
$$

where ϕ and ψ are the quadratic and cubic characters. Special choices of A and B could lead to evaluations such as (1.2).

The analogy between Gauss sums and gamma functions has been made several times previously [5, 9, 10, 12]. A systematic study of the properties of the Gaussian hypergeometric function is [8]. The properties of these functions should have implications for representation theory [11] and the geometry of curves [10].

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