Periodic solutions to some
difference equations over the integers.

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Abstract. In this paper, we investigate periodic integer solutions $\{a_n\}$ to

$$a_n = \begin{cases} r(a_{n-1} + a_{n-2}), \text{ if } r(a_{n-1} + a_{n-2}) \text{ is an integer,} \\ a_{n-1} + a_{n-2}, \text{ otherwise,} \end{cases}$$

where r is a rational number. We show that solutions can only exist if $-1 \le r \le \frac{1}{2}$, and we give several infinite families of r's for which the above recurrence has periodic solutions in the integers.

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1. Introduction and main theorems

It is known [1], that the difference equation

$$(1)\qquad a_n = \begin{cases} \dfrac{a_{n-1} + a_{n-2}}{2}, & \text{if } 2|a_{n-1} + a_{n-2}, \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases}$$

with a_0 and a_1 positive integers has the property that either a_n is stationary or it is unbounded. It is conjectured [8] that all solutions to

$$(2)\qquad a_n = \begin{cases} \dfrac{a_{n-1} + a_{n-2}}{3}, & \text{if } 3|a_{n-1} + a_{n-2}, \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases}$$

are unbounded except for certain obvious periodic solutions, such as the solutions 1, 1, 2, 1, 1, 2, … or 7, 14, 7, 7, 14, 7, …. Certainly for such difference equations, any solution which is not eventually periodic must be unbounded, so the real problem is to classify the periodic solutions to such an equation. This appears to be very difficult.

Systems such as (1) or (2) can be generalized in many ways. In [3], periodic solutions to $a_n = \lceil ca_{n-1} \rceil - a_{n-2}$ are studied for various real c. See [7] and [8] for other generalizations. Here, we consider another generalization of (2), and study the system

$$(3)\qquad a_n = \begin{cases} r(a_{n-1} + a_{n-2}), & \text{if } r(a_{n-1} + a_{n-2}) \text{ is an integer,} \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases}$$

where r is some fixed rational number. Again, it is very difficult to characterize those solutions to (3) which are periodic. Instead, we address a different, easier question:

Can we find other values of r which allow periodic solutions?

For example, if $r = \frac{1}{5}$, the initial conditions $a_0 = 1$, $a_1 = 1$ lead to the periodic solution 1, 1, 2, 3, 1, 4, 1, 1, …. In fact, the only positive r for which periodic solutions are known to exist are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$. Although we found no other positive values of r, we do have the following result.

Theorem 1.1 If $r > 0$ and the difference equation

$$a_n = \begin{cases} r(a_{n-1} + a_{n-2}), \text{ if } r(a_{n-1} + a_{n-2}) \text{ is an integer,} \\ a_{n-1} + a_{n-2}, \text{ otherwise,} \end{cases}$$

has a periodic solution, then $r = \frac{1}{N}$ for some integer $N \geq 2$. In particular, $0 < r \leq \frac{1}{2}$.

At present over all $r > 0$, only finitely many periodic orbits (r, a_0, a_1) with (a_0, a_1) relatively prime positive integers are known.

Next, we consider cases with $r \leq 0$. The case $r = 0$ in (3) is uninteresting so we suppose $r < 0$. We have the following complement to Theorem 1.1.

Theorem 1.2 If $r < 0$ and the difference equation

$$a_n = \begin{cases} r(a_{n-1} + a_{n-2}), \text{ if } r(a_{n-1} + a_{n-2}) \text{ is an integer,} \\ a_{n-1} + a_{n-2}, \text{ otherwise,} \end{cases}$$

has a periodic solution, then $-1 \leq r < 0$.

The lower bound $r = -1$ is sharp. In this case, the equation (3) becomes a linear equation, and all nontrivial solutions to (3) are periodic with period 3. In particular, there are infinitely many $(-1, a_0, a_1)$ with (a_0, a_1) relatively prime giving a periodic orbit. Combining this observation with Theorem 1.1 and Theorem 1.2, the only allowable values of r satisfy $-1 \leq r \leq \frac{1}{2}$ and these bounds are sharp.

The main results of this paper concern constructions of periodic orbits that establish the following result.

Theorem 1.3 There are infinitely many $-1 < r < 0$ for which the difference equation

$$a_n = \begin{cases} r(a_{n-1} + a_{n-2}), \text{ if } r(a_{n-1} + a_{n-2}) \text{ is an integer,} \\ a_{n-1} + a_{n-2}, \text{ otherwise,} \end{cases}$$

has a periodic solution.

Our constructions are of two kinds: A number of isolated cases and several infinite families. A summary of the infinite families is given in Table 5.8. In our search, the smallest value of $r > -1$ we have found which leads to periodic solutions is $r = -\frac{3}{4}$, with $a_0 = a_1 = 1$, and period length 16. The negative value of r we found closest to 0 which allows a periodic solution is $r = -\frac{1}{17}$, with initial conditions $a_0 = a_1 = 1$, and period 24. We do not know if either $-\frac{3}{4}$ or $-\frac{1}{17}$ are best possible for nonlinear equations (3).

One feature of our constructions is that for each fixed r we construct only finitely many different orbits (r, a_0, a_1) with (a_0, a_1) being relatively prime. We make the following conjecture.

Conjecture 1.4 For any specific value of r, with $-1 < r \leq \frac{1}{2}$, the difference equation

$$a_n = \begin{cases} r(a_{n-1} + a_{n-2}), \text{ if } r(a_{n-1} + a_{n-2}) \text{ is an integer,} \\ a_{n-1} + a_{n-2}, \text{ otherwise,} \end{cases}$$

has only finitely many relatively prime initial conditions giving purely periodic solutions.

In this regard, the difference equation (3) is reminiscent of the Colatz problem, which is given by a first order non−homogeneous difference equation

$$a_n = \begin{cases} \frac{a_{n-1}}{2}, \text{ if } \frac{1}{2}a_{n-1} \text{ is an integer,} \\ 3a_{n-1} + 1, \text{ otherwise.} \end{cases}$$

This equation is conjectured to have only one periodic orbit for positive integer a_0, and more generally to have only finitely many periodic orbits over all integers. See [9] for this conjecture and for a general survey on this problem and related problems.

Our approach to finding periodic solutions to (3) is to convert the problem from a second order equation to a first order system. If a periodic solution to (3) exists, then there is a matrix, depending on the period, which has 1 as an eigenvalue. We performed a brute force search for such matrices. Many patterns became obvious in that search. Theorem 1.3 is verified by justifying these patterns. The theoretical details of our approach are discussed in detail in Section 2.

In Section 3 we prove Theorem 1.1 and Theorem 1.2. In Section 4 we give a sufficient eigenvector condition on a first order system to give a periodic orbit. This condition is applied in later sections in our constructions. In Section 5 we give explicit constructions, delaying until Section 6 details of the construction of some infinite families. Theorem 1.3 follows as a direct consequence of the constructions in Section 5, or by Theorem 6.2. In Section 7 we make concluding remarks on finding more periodic orbits.

All the infinite families we find are describable in terms of Fibonacci and Lucas numbers. We use F_n to refer to the Fibonacci numbers (with initial conditions $F_0 = 0$, $F_1 = 1$), and L_n to refer to the Lucas numbers: $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Our results lead to some new Fibonacci number identities given in Section 7.

2. Difference equations and matrix products.

As is well–known, second order difference equations can be converted into first order systems of difference equations. For the general second order system (3), an associated first order system is the following:

For vectors $\mathbf{v}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, with integer entries,

$$\text{(4)} \qquad \mathbf{v}_n = \begin{cases} B\mathbf{v}_{n-1}, \text{ if } r(x_{n-1} + y_{n-1}) \text{ is an integer,} \\ A\mathbf{v}_{n-1}, \ \text{ otherwise,} \end{cases}$$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $B = \begin{pmatrix} r & r \\ 1 & 0 \end{pmatrix}$. We may replace the condition that $r(x_{n-1} + y_{n-1})$ is an integer by the equivalent matrix product condition that $((1, 1)\ \mathbf{v_{n-1}})\ r$ be an integer.

For example, using $r = \frac{1}{5}$, the periodic solution given in Section 1 corresponds to the following periodic solution to (4): $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ldots.$ In this formulation, $\mathbf{v}_n = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$, where the a's are in terms of (3).

Given an initial vector $\mathbf{v_0}$, every subsequent $\mathbf{v_n}$ can be obtained from $\mathbf{v_0}$ via a formula $\mathbf{v_n} = M_n\mathbf{v_0}$, where M_n is an appropriate product of n A's and B's. Thus, for the example above, if $\mathbf{v_0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $\mathbf{v_1} = A\mathbf{v_0}$, $\mathbf{v_2} = A\mathbf{v_1} = A^2\mathbf{v_0}$, $\mathbf{v_3} = BA^2\mathbf{v_0}$, $\mathbf{v_4} = ABA^2\mathbf{v_0}$, and so on. Note that the matrices multiply $\mathbf{v_0}$ from right to left. That is, to obtain $\mathbf{v_4}$, $\mathbf{v_0}$ is multiplied by A, then A, then B, then A. Note also that $\mathbf{v_6} = \mathbf{v_0}$, so $B^2ABA^2\mathbf{v_0} = \mathbf{v_0}$. As a consequence, we have the following:

Theorem 2.1. If (4) has a periodic solution of period k, and $\mathbf{v_0}$ is a vector in that periodic solution, then there is a corresponding matrix M, which is a product of k A's and B's, and $\mathbf{v_0}$ is an eigenvector of M with eigenvalue 1.

While the proof of Theorem 2.1 is obvious, we mention that the converse is not true. The obstruction is that the sum of the entries in some $\mathbf{v_j}$ might be divisible by the denominator of r at a time when multiplication by A is called for. For example, if $r = -\frac{1}{5}$, then $M = A^{11}BA^2BA^5BA^2B$ has $\mathbf{v} = \begin{pmatrix} 34 \\ 21 \end{pmatrix}$ as an eigenvector with eigenvalue 1, but the sequence which starts 21, 34 continues −11, 23, 12, −7, −1, −8, −9, −17, ..., with the terms decreasing monotonically to $-\infty$. Here, $BA^2B\,\mathbf{v} = \begin{pmatrix} -7 \\ 12 \end{pmatrix}$ has a sum of entries divisible by 5, so (4) calls for a multiplication by B, but the form of M indicates a multiplication by A.

When the form of M does not conform to the rules in (4), there are three possibilities: First, as above, the sequence might be aperiodic. Second, it is possible that the initial vector $\mathbf{v_0}$ might belong to a preperiod. For example, if $r = -\frac{1}{3}$ and $M = A^8BA^4BA^3B$, then M has $\begin{pmatrix} 11 \\ 7 \end{pmatrix}$ as an eigenvector with eigenvalue 1 and the sequence beginning 7, 11 eventually enters the cycle 0, −1, −1, −2, 1, −1. Finally, it is possible that $\mathbf{v_0}$ might belong to a periodic solution, but with a different M. For example, if $r = -\frac{1}{3}$, then $M = A^5B$ has $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ as an eigenvector with eigenvalue 1. Here, the sequence which starts 1, 2 continues −1, 1, 0, 1, 1, 2, $\cdots$, which is periodic, but corresponds to the matrix A^3BAB rather than A^5B. This can happen if a vector $\mathbf{v_j}$ has the form $\begin{pmatrix} -a \\ a \end{pmatrix}$, in which case $A\mathbf{v_j} = B\mathbf{v_j}$.

As a consequence of Theorem 2.1 and the remarks above, to find values of r for which system (3) will admit a periodic solution, one approach is as follows:

1. Search for a product of matrices A and B that has 1 as an eigenvalue,
2. Find a resulting eigenvector, scaled so as to have relatively prime integer entries, with bottom component nonnegative,
3. Check that this eigenvector actually corresponds to a periodic solution.

We list the following linear algebra facts (see, for example, [11, Chap.7]), followed by the form most useful to us.

Theorem 2.2.

a. If v is an eigenvector with eigenvalue λ for a matrix M, then v is an eigenvector with eigenvalue λ^k for M^k.

b. The characteristic polynomial for a 2×2 matrix M has the form

$$\chi_M(z) = \det(zI - M) = z^2 - \operatorname{tr}(M)\, z + \det(M),$$

where tr(M) is the trace of M, the sum of the diagonal entries of M.

c. The characteristic polynomials for MN and NM are the same for any n×n matrices M and N. In particular, tr(MN) = tr(NM).

d. The trace is linear. That is, tr(cM + dN) = c tr(M) + d tr(N), for any n×n matrices M, N and scalars c, d.

e. Every matrix satisfies its characteristic polynomial. In particular, for 2×2 matrices M, $M^2 - \operatorname{tr}(M)\, M + \det(M)I = 0$.

Corollary 2.3

a. If v is an eigenvector with eigenvalue 1 for a matrix M, then v is an eigenvector with eigenvalue 1 for M^k.

b. If v is an eigenvector with eigenvalue −1 for a matrix M, then v is an eigenvector with eigenvalue 1 for M^{2k}.

c. A 2×2 matrix M has 1 as an eigenvalue if and only if

$$\det(I - M) = 1 - \operatorname{tr}(M) + \det(M) = 0.$$

d. If $M_1M_2\cdots M_k$ has 1 as an eigenvalue, then so does a cyclic permutation, $M_kM_1M_2\cdots M_{k-1}$.

Also of use to us is the following theorem:

Theorem 2.4 If M is a product of some number of 2×2 matrices A and B, say $M = N_1N_2\ldots N_k$, where each N_i is either A or B, then M has 1 as an eigenvalue if and only if $M' = N_kN_{k-1}\cdots N_1$ has 1 as an eigenvalue.

We call M' the reversal of M.

Sketch of Proof. We need only show one direction of the proof, so suppose M has 1 as an eigenvalue. Then $1 - \operatorname{tr}(M) + \det(M) = 0$. Since $\det(M) = \det(M')$, it is enough to show that $\operatorname{tr}(M) = \operatorname{tr}(M')$. For this, we induct on k, the result being trivial when k = 1, and the k = 2 case follows from Theorem 2.2(c).

Suppose the result of the theorem is true for products of fewer than k matrices. Then given $M = N_1N_2\ldots N_k$, if $N_i = N_{i+1}$ for any i then one may use Theorem 2.2 (d) and (e) to write M as a linear combination of fewer products and invoke the induction hypothesis. If M does not contain two consecutive A's or B's, then the A's and B's alternate. Products ABAB…BA or BABA…AB are their own reversals so they have the same trace. Finally, for ABAB…AB or BABA…BA, the reversal is also a cyclic permutation so by Corollary 2.3 (d), M and M′ have the same trace. □

Corollary 2.3(d) can be interpreted as analogous to the fact that if a sequence, say 1, 2, 3, 4, 5 is a periodic solution to (3), then any cyclic shift such as 3, 4, 5, 1, 2 is also a periodic solution. Theorem 2.4 does not appear to have such an interpretation. For example, by Theorem 2.4, given that A^2BAB^2 has 1 as an eigenvalue when $r = 1/5$, we know that B^2ABA^2 also has one as an eigenvalue. For the first matrix, an eigenvector is $\begin{pmatrix} 11 \\ 9 \end{pmatrix}$, corresponding to the periodic solution 9, 11, 4, 3, 7, 2, $\cdots$ for (3), but for the second matrix, an eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, corresponding to the periodic solution 1, 1, 2, 3, 1, 4.

We make use of the following facts about A^k.

Theorem 2.5. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then for all integers n and k,

a. $A^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$,

b. $A^k\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} F_{n+k+1} \\ F_{n+k} \end{pmatrix}$, $\qquad A^k\begin{pmatrix} L_{n+1} \\ L_n \end{pmatrix} = \begin{pmatrix} L_{n+k+1} \\ L_{n+k} \end{pmatrix}$,

c. $A^k\begin{pmatrix} -F_n \\ F_{n+1} \end{pmatrix} = (-1)^k\begin{pmatrix} -F_{n-k} \\ F_{n+1-k} \end{pmatrix}$, $\qquad A^k\begin{pmatrix} -L_n \\ L_{n+1} \end{pmatrix} = (-1)^k\begin{pmatrix} -L_{n-k} \\ L_{n+1-k} \end{pmatrix}$,

where F_{-m} is defined to be $(-1)^{m+1}F_m$ and $L_{-m} = (-1)^m L_m$.

Proof. These results are all easy inductions. The first parts of (b) and (c) also follow by considering the first column in the product A^kA^n and the last column in A^kA^{-n}. □

Fibonacci identities come up frequently in this paper. Long lists of such identities can be found in many places, including on the web, say in [2] or [5]. We mention two approaches to proving such identities other than induction: First one may always use formulas such as

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n.$$

Alternatively, any combination of products of Fibonacci numbers (or Lucas numbers) will satisfy a d−th order linear recurrence for some d. Given a proposed identity of the form

$$\text{Expression}_n = 0,$$

find the order, d, of Expression_n, and check that $\text{Expression}_n = 0$ for n = 0, 1, 2, …, d−1. Thus, for a rigorous proof of Cassini's identity:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \tag{5}$$

one legitimate approach would be to show that the order, d, is 3 and check that the formula holds for the cases n = 0, 1, 2. See [12] for a discussion on how to prove many algebraic identities simply by verifying that they hold for specific, well chosen values of the parameters. We will predominantly use this second method here so as to avoid proving or looking up identities that we require.

One more result on sequences which obey the Fibonacci relation is needed.

Theorem 2.6. Suppose $\{a_n\}$ satisfies the Fibonacci relation, $a_k = a_{k-1} + a_{k-2}$, for all k.

a. If $a_j > 0$ and $a_{j+1} > 0$, then a_{j+k} is a monotonically increasing sequence for $k \geq 1$.

b. If $(-1)^j a_j > 0$ for $k \leq j \leq k+N$, then $|a_{j+k}|$ is a monotonically decreasing sequence for $0 \leq k \leq N - 1$.

c. If $a_j = 0$ for some j, then for some constant c, $a_k = cF_{k-j}$ for all k.

Proof. The result in (a) is trivial. For (b), without loss of generality, we may take k to be 0. Let $b_j = (-1)^{N-j}a_{N-j}$ for all j. Then for $0 \leq j \leq N$, the b's are positive, and satisfy the recurrence $b_j = -b_{j+1} + b_{j+2}$, or $b_{j+2} = b_{j+1} + b_j$. Thus, the b's are increasing from j = 1, meaning the a's are decreasing in absolute value to N−1. Finally, given that $a_j = 0$, let $a_{j+1} = c \neq 0$. If $b_k = \frac{1}{c}a_{k-j}$, then $b_0 = 0$, $b_1 = 1$, and the b's satisfy the Fibonacci relation. Hence, $b_k = F_k$ for all k, and the result follows. □

We now proceed as follows: Let $B = \begin{pmatrix} x & x \\ 1 & 0 \end{pmatrix}$. Given $M = N_1N_2\ldots N_k$, where each N_i is either A or B, we form the polynomial

$$f_M(x) = -\det(I - M) = -1 + \text{tr}(M) - \det(M), \tag{6}$$

and seek M for which $f_M(x)$ has rational zeros. With the negative sign, most of the coefficients of $f_M(x)$ are positive integers. Using Corollary 2.3 (d), we need only look at equivalence classes of products corresponding to cyclic permutations of the matrices whose product is M. These equivalence classes are often called necklaces. For example, A^2BAB^2, BA^2BAB, B^2A^2BA, AB^2A^2B, and BAB^2A^2 all belong to the same necklace, all generate the same polynomial $f_M(x)$, and for $x = \frac{1}{5}$, all generate essentially the same periodic solution to (3). We standardize our necklaces to have the longest run of A's on the left hand side. Thus in the example above, we only consider A^2BAB^2. In general, when performing a brute force search of all products of n A's and B's, restricting to necklaces cuts the number of cases by roughly a factor of n.

3. **Existence properties for system (3)**

In this section, we derive properties of $f_M(x)$, defined in (6). These properties of $f_M(x)$ are used to derive restrictions on values of x that allow periodic solutions to (3). In particular, the proofs of Theorem 1.1 and Theorem 1.2 will follow from facts about $f_M(x)$.

Lemma 3.1. If $f_M(r) = 0$ then for every $k > 0$, $f_{M^k}(r) = 0$.

Proof. We have $f_{M^k}(x) = -\det(I - M^k) = -\det(I - M)\det(M^{k-1} + M^{k-2} + \cdots + I)$. Thus, $f_{M^k}(x) = f_M(x)\, q(x)$ for some polynomial $q(x)$, and the result follows. □

Of particular use in the proof of Theorem 1.2 and in Section 5 will be the case $f_{M^2}(x) = f_M(x)\det(I + M)$. Three additional lemmas are required in the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 3.2. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} x & x \\ 1 & 0 \end{pmatrix}$, then for $n_i \geq 0$, $\mathrm{tr}(A^{n_1}BA^{n_2}B\cdots A^{n_k}B)$ is a polynomial in x of degree k with nonnegative integer coefficients, with constant term $F_{n_1}F_{n_2}\cdots F_{n_k}$ and leading coefficient $F_{n_1+2}\,F_{n_2+2}\cdots F_{n_k+2}$.

Proof. By Theorem 2.5 (a), we have

$$A^nB = \begin{pmatrix} F_{n+1}x+F_n & F_{n+1}x \\ F_nx+F_{n-1} & F_nx \end{pmatrix} = x\begin{pmatrix} F_{n+1} & F_{n+1} \\ F_n & F_n \end{pmatrix} + \begin{pmatrix} F_n & 0 \\ F_{n-1} & 0 \end{pmatrix}.$$

Thus,

$$A^{n_1}BA^{n_2}B\cdots A^{n_k}B$$
$$= x^k\begin{pmatrix} F_{n_1+1} & F_{n_1+1} \\ F_{n_1} & F_{n_1} \end{pmatrix}\cdots\begin{pmatrix} F_{n_k+1} & F_{n_k+1} \\ F_{n_k} & F_{n_k} \end{pmatrix} + \cdots + \begin{pmatrix} F_{n_1} & 0 \\ F_{n_1-1} & 0 \end{pmatrix}\cdots\begin{pmatrix} F_{n_k} & 0 \\ F_{n_k-1} & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} ac & 0 \\ bc & 0 \end{pmatrix}$, the constant term will be $F_{n_1}F_{n_2}\cdots F_{n_k}$.

Similarly, for the leading coefficient, we may induct on the simple calculation $\begin{pmatrix} a & a \\ b & b \end{pmatrix}\begin{pmatrix} c & c \\ d & d \end{pmatrix} = (c + d)\begin{pmatrix} a & a \\ b & b \end{pmatrix}$ to show that the trace of the product is the product of the traces. By the Fibonacci relation, the trace of each individual matrix is F_{n_i+2}. □

Lemma 3.3. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} x & x \\ 1 & 0 \end{pmatrix}$. If $m \geq 1$, $n \geq 1$ and $x \leq -1$, then

$$A^mBA^nB = P + x^2\, Q,$$

where P is a matrix with nonnegative entries and Q is either I or $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$.

Proof. This result follows by induction on m + n. To start the induction, we note that

$$ABAB = \begin{pmatrix} x^2 + 2x + 1 & 2x^2 + x \\ 2x^2 + x & x^2 \end{pmatrix} + x^2I, \qquad A^2BAB = \begin{pmatrix} 3x^2 + 3x + 1 & 4x^2 + x \\ 2x^2 + 2x + 1 & x^2 + x \end{pmatrix} + x^2I,$$
$$A^2BA^2B = \begin{pmatrix} 5x^2 + 6x + 1 & 6x^2 + 2x \\ 3x^2 + 4x + 1 & 2x^2 + 2x \end{pmatrix} + x^2I, \quad ABA^2B = \begin{pmatrix} 3x^2 + 4x + 1 & 3x^2 + x \\ 3x^2 + x & x^2 \end{pmatrix} + x^2\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

The 16 polynomials in the above matrices are all nonnegative for $x \leq -1$. For the inductive step, we assume the result is true if $m + n < N$, and use $A^k = A^{k-1} + A^{k-2}$ if either m or n is greater than 3 to reduce to that case. □

Lemma 3.4. If $M = A^{n_1}BA^{n_2}B\cdots A^{n_k}B$, where there are $n = k + n_1 + n_2 + \cdots + n_k$ matrices in the product, then

$$f_M(x) = (F_{n_1+2}F_{n_2+2}\cdots F_{n_k+2} - (-1)^n)x^k + (\text{terms with nonnegative coefficients})$$
$$+ F_{n_1}F_{n_2}\cdots F_{n_k} - 1.$$

Proof. This follows from Lemma 3.2 since $\det(A) = -1$ and $\det(B) = -x$, so

$$\det(M) = (-1)^{\#\text{ of A's}}\times(-x)^{\#\text{ of B's}} = (-1)^n\, x^k. \qquad \square$$

Proof of Theorem 1.1. Let $M = A^{n_1}BA^{n_2}B\cdots A^{n_k}B$, where there are $n = k + n_1 + n_2 +$

$\cdots + n_k$ matrices in the product. If all the n's are zero then $M = B^k$. The only x–values for B^k which allow 1 as an eigenvalue are $x = -1$ and $x = \frac{1}{2}$ because the characteristic polynomial for B has no roots on the unit circle in the complex plane except for these values of x.

If any of the n's is positive, $F_{n_1+2}\, F_{n_2+2} \cdots F_{n_k+2} - (-1)^n > 0$. Since the coefficients of $x, x^2, \ldots, x^{k-1}$ are all nonnegative, $f_m(x)$ can only have a positive zero if the constant term is negative. By Lemma 3.4, the smallest the constant term can be is -1, occurring if some n is zero. By the Rational Root Theorem [10, Theorem 4–9, p. 148], any rational solution must then be of the form $\frac{1}{b}$, where b is a divisor of $F_{n_1+2}\, F_{n_2+2} \cdots F_{n_k+2} - (-1)^n$. □

Proof of Theorem 1.2. Again let $M = A^{n_1}BA^{n_2}B\cdots A^{n_k}B$, where there are $n = k + n_1 + n_2 + \cdots + n_k$ matrices in the product. Let r be a zero of $f_M(x)$ with $r < 0$. By Lemma 3.1, r is also a zero of $f_{M^2}(x)$. As a consequence, we may assume that both n and k are even.

As in the proof of Theorem 1.1, if any of the n's is zero, then the constant term in $f_M(x)$ is -1, and the only rational zeros are reciprocals of integers. Thus, $-1 \le r < 0$ in this case. Suppose next that all the n's are positive and by way of contradiction, suppose that $r < -1$. Since k is even we may partition M into products of the form A^mBA^nB. Letting $k = 2j$, by Lemma 3.3,

$$\begin{aligned} M &= (A^{n_1}BA^{n_2}B)\cdots(A^{n_{2j-1}}BA^{n_{2j}}B) \\ &= (P_1 + r^2Q_1)(P_2 + r^2Q_2)\cdots(P_j + r^2Q_j), \end{aligned}$$

where the P's have nonnegative entries, and each Q is either I or $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. Hence,

$$M = U + r^{2j}\, Q',$$

where U has nonnegative entries and Q' has a trace of 2. Thus,

$$\begin{aligned} f_M(r) &= -1 + \mathrm{tr}(M) - \det(M) \\ &= \mathrm{tr}(U) + 2r^{2j} - 1 - r^{2j} \;=\; \mathrm{tr}(U) + r^{2j} - 1. \end{aligned}$$

Since $\mathrm{tr}(U) \ge 0$, and $r^2 > 1$, $f_M(r) > 0$, contradicting $f_M(r) = 0$. □

We give an additional theoretical result. Our numerical data suggested that more products M of A's and B's gave polynomials $f_M(x)$ having rational zeros r when the number of matrices was divisible by 3 than otherwise. The following theorem supplies an explanation.

Theorem 3.5. If $r = \frac{p}{q}$ in (3), where p and q are both odd, then any periodic solution must have period length divisible by 3.

Proof. Given a periodic solution, we may divide by common factors of 2 to produce a sequence with initial values x_0, x_1, with at least one of them odd. Consequently, there is a matrix M so that $M\begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}$, and M is the product of n A's and B's, where n is the length of the period. We consider this equation modulo 2. Since $r \equiv 1 \pmod 2$, $B \equiv A$, so $M \equiv A^n \pmod 2$. Thus, A^n must have an eigenvector modulo 2, but this only occurs when n is divisible by 3. □

4. A Sufficient Condition for a Periodic Orbit

The following theorem is the main tool we used to prove that various infinite families of r produce periodic solutions to (3).

Theorem 4.1. Let $r = \frac{p}{q}$, where the fraction is in lowest terms, with $q > 0$. Suppose that $M = A^{n_1}BA^{n_2}B\cdots A^{n_k}B$ and $\mathbf{v}$ is a nontrivial integer–valued eigenvector with eigenvalue 1 or −1 for M. Let $\mathbf{v_0} = \mathbf{v}$, $\mathbf{v_1} = A^{n_k}B\,\mathbf{v}$, $\mathbf{v_2} = A^{n_{k-1}}BA^{n_k}B\,\mathbf{v}$, …, $\mathbf{v_{k-1}} = A^{n_2}B\cdots A^{n_k}B\,\mathbf{v}$. If

(a) $(1, 1)\,v$ is a multiple of q,

(b) for each $i \geq 1$, $(1, 1)\,\mathbf{v_i} = 0$ or $\pm q$,

(c) for each $i \geq 1$, $|(1, 1)\,B\mathbf{v_i}| < q$,

and

(d) $(1, 1)A^mB\mathbf{v_{k-1}}$ is not divisible by q for $0 \leq m < n_1$,

then v corresponds to a periodic solution to (3). The length of the period is $n = k + n_1 + n_2 + \cdots + n_k$ if $\mathbf{v}$ has eigenvalue 1, and 2n if $\mathbf{v}$ has eigenvalue −1.

In most of the cases where we invoke Theorem 4.1, $(1, 1)\,\mathbf{v} = q$ in condition (a), which renders condition (d) trivial. It is also often the case that both coordinates of each $\mathbf{v_i}$ have the same sign. In this case, if $r < 0$, condition (c) follows from condition (b). That is, if $\mathbf{v_i} = \begin{pmatrix} a \\ b \end{pmatrix}$ where, say a and b are positive and $a + b = q$, then $(1, 1)Bv_i = a + p$. Thus $|a + p| < q$ since $p < 0$, and $|p| < q$.

Proof. To show that **v** corresponds to a periodic solution to (3), we show that steps involving multiplication by r only occur "at the right times." That is, only at the beginning, after n_k+1 steps, and so on. We will allow one exception to this as explained below. For a fixed i, let $u_j = (1, 1)A^jB\mathbf{v_i}$. That is, we assign a sequence of u's to each $\mathbf{v_i}$. We show that u_j is not divisible by q unless $j = n_{k-i}$. (For this j, $A^jB\mathbf{v_i} = \mathbf{v_{i+1}}$.) Condition (d) handles the case where $i = k-1$. We assume $i < k-1$ in the remainder. Since $A^j = A^{j-1} + A^{j-2}$, $\{u_j\}$ satisfies the Fibonacci relation. By condition (c), $|u_0| < q$. If p is positive, then all the u's are positive, increasing. Thus, $u_{n_{k-i}} = q$ and $0 < u_j < q$ for $0 \le j \le n_{k-i} - 1$. If p is negative, it is possible that the u's alternate in sign for a time before becoming monotonic. By Theorem 2.6, while alternating in sign, the absolute value of u_j decreases. If $u_{n_{k-i}} = 0$, then $0 < |u_j| < q$ for $0 \le j \le n_{k-i} - 1$. Otherwise, once the terms become monotonic, the absolute value increases, but $|u_{n_{k-i}}| = q$. Thus, $|u_j| < q$ for $0 \le j \le n_{k-i} - 1$. If $u_j \ne 0$ for $0 \le j \le n_{k-i} - 1$, we are done. The only possible obstruction is that $u_j = 0$ for some j. Then one should multiply by B rather than by A at this stage. However, if $u_j = 0$, then $A^jB\mathbf{v_i} = \begin{pmatrix} -c \\ c \end{pmatrix}$ for some c. Since $B\begin{pmatrix} -c \\ c \end{pmatrix} = A\begin{pmatrix} -c \\ c \end{pmatrix}$, the original vector **v** is also an eigenvector for the matrix M' formed by changing the appropriate A to a B. Since the u's decrease in absolute value until they become monotone, and increase in absolute value from there, there can be at most one j between 0 and n_{k-i} with $u_j = 0$. Thus, after replacing at most k A's with B's, we never have a $u_j = 0$ except at $j = n_{k-i}$. This completes the proof in the case where v has eigenvalue 1. If **v** has eigenvalue -1, then since $M\mathbf{v} = -\mathbf{v}$, it is clear that **v** will be part of a periodic solution with period 2n. □

As an example of the proof above, consider $M = A^9B$, $r = -\frac{3}{8}$. The matrix M has $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ as an eigenvector. The sequence of u's is 2, −1, 1, 0, 1, 1, 2, 3, 5, 8. Since $u_3 = 0$, a multiplication by B was called for at some point rather than A. Changing the appropriate A to a B gives $M' = A^5BA^3B$, which has the same eigenvector.

5. Infinite families of periodic solutions

Using MATHEMATICA, we checked all products of up to 24 matrices, finding all rational zeros of $f_M(x)$ for these cases. We used MAPLE to find the eigenvectors for each such M, and then checked to see if the eigenvectors corresponded to periodic solutions to (3). Based on these calculations, we found many patterns, which we proved to hold in general. We organize our results below by the number of B's in the product (or equivalently, by the degree of $f_M(x)$). We also categorize results based on the shortest period. That is, given, say, that A^2BAB^2 allows $r = \frac{1}{5}$, we ignore $r = \frac{1}{5}$ for $(A^2BAB^2)^2$.

We observed the following: For a given fixed number of B's, there appear to be a finite number of "sporadic" cases along with some (possibly 0) infinite families where $f_M(x)$ has rational zeros. We now proceed by cases, starting with a single B.

5.1 **One B**

With one B, there is always an r−value since $M = A^nB$ has

$$f_M(x) = (F_{n+2} - (-1)^{n+1})x + F_n - 1.$$

The zero of $f_M(x)$ is $-\dfrac{F_n - 1}{F_{n+2} + (-1)^n}$, with corresponding eigenvector $\mathbf{v} = \begin{pmatrix} F_{n+1} \\ F_n + (-1)^n \end{pmatrix}$.

Noting that $(1, 1)\mathbf{v} = F_{n+1} + F_n + (-1)^n$ is the denominator of r, by Theorem 4.1, this always corresponds to a periodic solution to (3) provided **v** is an eigenvector. To show **v** is an eigenvector, we show $A^nB\mathbf{v} = \mathbf{v}$. To this end, $B\mathbf{v} = \begin{pmatrix} 1 - F_n \\ F_{n+1} \end{pmatrix}$, so

$$A^nB\mathbf{v} = A^n\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -F_n \\ F_{n+1} \end{pmatrix}\right) = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} + (-1)^n\begin{pmatrix} -F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n + (-1)^n \end{pmatrix} = \mathbf{v},$$

as desired. Here, we have used Theorem 2.5 (c) as part of the calculation.

By Theorem 4.1, we know that the periodic solution to (3) here will involve one, or maybe two multiplications by r, depending on whether any u_j of Theorem 4.1 is 0. It turns out that there is one multiplication by r unless n has the form 4m+1. To see this, we calculate

$$u_j = (1, 1)A^jB\mathbf{v} = (1, 1)A^j\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -F_n \\ F_{n+1} \end{pmatrix}\right) = (1, 1)\left(\begin{pmatrix} F_{j+1} \\ F_j \end{pmatrix} + (-1)^j\begin{pmatrix} -F_{n-j} \\ F_{n+1-j} \end{pmatrix}\right)$$

$$= F_{j+2} + (-1)^jF_{n-1-j}.$$

For u_j to be 0, j must be odd, and $j+2 = n-j-1$, so $n = 2j+3 = 2(2m-1) + 3 = 4m + 1$. In this case, where $n = 4m + 1$, $u_j = 0$ when $j = 2m-1$, we have proven that $\begin{pmatrix} F_{4m+2} \\ F_{4m+1} - 1 \end{pmatrix}$ is an eigenvector, with eigenvalue 1 for $A^{2m+1}BA^{2m-1}B$, when $r = -\dfrac{F_{4m+1} - 1}{F_{4m+3} - 1}$, and this corresponds to a periodic solution to (3) in this case. A simplification is possible here: $F_{4m+2} = F_{2m+1}L_{2m+1}$, $F_{4m+1} - 1 = F_{2m}L_{2m+1}$, $F_{4m+3} - 1 = F_{2m+2}L_{2m+1}$, which means that $\begin{pmatrix} F_{2m+1} \\ F_{2m} \end{pmatrix}$ is an eigenvector for $A^{2m+1}BA^{2m-1}B$, when $r = -\frac{F_{2m}}{F_{2m+2}}$.

If n has the form 4m+3, although there is only one multiplication by B, the form of r and the eigenvector can be simplified to $r = -\frac{L_{2n-1}}{L_{2n+1}}$, $\mathbf{v} = \begin{pmatrix} L_{2n} \\ L_{2n-1} \end{pmatrix}$. No further simplification is possible in these cases because any two Fibonacci or Lucas numbers with index differing by 1 or 2 are relatively prime [6, Theorem A, page 80]. We point out that based on these cases, we have established the proof of Theorem 1.3. That is, we have given periodic solutions to (3) for all r of the form $-\frac{F_{2m}}{F_{2m+2}}$ or $-\frac{L_{2n-1}}{L_{2n+1}}$.

5.2 **Two B's**

In our search for periodic solutions with two B's, we found one isolated case and two infinite families, as shown in the table below.

Table 5.1

M	r	eigenvector
A^2BAB	$-\frac{4}{7}$	$\begin{pmatrix} 9 \\ 5 \end{pmatrix}$
A^nBA^nB	$-\frac{F_n+1}{F_{n+2}-(-1)^n}$	$\begin{pmatrix} F_{n+1} \\ F_n-(-1)^n \end{pmatrix}$
$A^{2n+1}BA^{2n-1}B$	$-\frac{F_{2n}}{F_{2n+2}}$	$\begin{pmatrix} F_{2n+1} \\ F_{2n} \end{pmatrix}$

The first line is, of course, just a simple calculation. We did not list two cases, one where r = 0, and another where r = −1 since these do not correspond to nonlinear systems (3).

The third line of the table comes from our previous work. The middle line of the table corresponds to finding an eigenvector with eigenvalue −1 for A^nB. The proof that the given eigenvector works is nearly identical to the proof when the eigenvalue was 1. When n = 4m+3, the u_j of Theorem 4.1 will be 0 for an appropriate j, yielding a line in a table for four B's (specifically, the family $(A^{2n+2}BA^{2n}B)^2$). Also, as in the case of one B, when n is odd, the value of r and the eigenvector can be simplified by canceling common factors. The simplifications give $r = -\frac{F_{2n+1}}{F_{2n+3}}$, $\mathbf{v} = \begin{pmatrix} F_{2n+2} \\ F_{2n+1} \end{pmatrix}$, and $r = -\frac{L_{2n}}{L_{2n+2}}$, $\mathbf{v} = \begin{pmatrix} L_{2n+1} \\ L_{2n} \end{pmatrix}$. Combining these with the results for a single B, we have that for all n, $r = -\frac{F_n}{F_{n+2}}$, $\mathbf{v} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$ and $r = -\frac{L_n}{L_{n+2}}$, $\mathbf{v} = \begin{pmatrix} L_{n+1} \\ L_n \end{pmatrix}$ give periodic solutions.

With more than two B's, we use the short hand $(n_1, n_2, \ldots, n_k)$ to refer to M = $A^{n_1}BA^{n_2}B\cdots A^{n_k}B$. For example, we denote A^2BAB^2 by (2, 1, 0). In this scheme, the number of terms in the sequence is the number of B's, and the sum of the terms is the number of A's. With products containing only one or two B's, reversals are part of the same necklace, so they do not lead to different periodic solutions. Beginning with k = 3, reversals matter. The reversal of (a, b, c) can be put in the form (a, c, b). In our standard format, $a \geq b$, $a \geq c$, so we need only consider reversals when $b \neq c$. More generally, the reversal of $(a_1, a_2, a_3, \ldots, a_k)$ is equivalent to $(a_1, a_k, a_{k-1}, \ldots, a_2)$.

5.3 Three B's

As the number of B's grows, the number of sporadic cases appears to increase. With three B's, we found 6 such sporadic cases (ignoring r = 0 and r = −1), but only two of them gave purely periodic solutions to (3). The two were $A^7BA^4BA^4B$, which had $r = -\frac{4}{15}$ and eigenvector $\begin{pmatrix} 47 \\ 28 \end{pmatrix}$, and $A^9BA^2BA^2B$, which had $r = -\frac{3}{4}$ and eigenvector $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$. In this second case, one of the u's was zero, so the sequence more properly corresponds to the matrix $A^5BA^3BA^2BA^2B$. In contrast, one of the cases that did not produce a purely periodic solution was $A^8BA^4BA^3B$, which has $\begin{pmatrix} 11 \\ 7 \end{pmatrix}$ as an eigenvector with eigenvalue 1 when $r = -\frac{1}{3}$, but the sequence that begins 7, 11 is not purely periodic, but is in the preperiod of an eventually periodic sequence.

The infinite families with k = 3 are:

Table 5.2

M	r	eigenvector	solution to (3)
(2n+1, n+1, n−1)	$-\frac{F_n}{F_{n+2}}$	$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$	no
(2n+1, n−1, n+1)	$-\frac{F_n}{F_{n+2}}$	$\mathbf{v_n}$	no?
(4n+4, 4n+2, 4n+3)	$-\frac{a_n}{b_n}$	$\begin{pmatrix} F_{4n+3}F_{2n+3} \\ F_{4n+4}F_{2n+1} \end{pmatrix}$	yes
(4n+4, 4n+3, 4n+2)	$-\frac{a_n}{b_n}$	?	yes?
(4n+2, 4n, 4n+1)	$-\frac{c_n}{d_n}$	$\begin{pmatrix} F_{4n+1}L_{2n+2} \\ F_{4n+2}L_{2n} \end{pmatrix}$	yes
(4n+2, 4n+1, 4n)	$-\frac{c_n}{d_n}$	?	yes?

In the table above, $$\mathbf{v_n} = \frac{1}{2}\begin{pmatrix} F_{n+2}^3 + F_n^3 + (-1)^n F_{n-1} \\ F_{n+2}^3 - F_n^3 - (-1)^n F_{n-1} \end{pmatrix},$$

$$a_n = F_{4n+4}F_{2n} + F_{4n+2}F_{2n+1}, \qquad b_n = F_{4n+4}F_{2n+1} + F_{4n+3}F_{2n+3},$$
$$c_n = F_{4n+1}L_{2n-1} + F_{4n}L_{2n+1}, \qquad d_n = F_{4n+2}L_{2n} + F_{4n+1}L_{2n+2}.$$

We comment that we have not proven that $\mathbf{v_n}$ is actually an eigenvector for $A^{2n+1}BA^{n-1}BA^{n+1}B$ though it fits the pattern for all n from 1 to 10. Also, we have not demonstrated that $\mathbf{v_n}$ never leads to a solution to (3), though it did not for those same values of n. We do not know of a general formula for the eigenvectors in the fourth and sixth rows. For small values of n, the eigenvectors found always corresponded to solutions to (3). We postpone the proof that rows three and five give infinite families of solutions to (3) to the Section 6.

5.4 **Four B's**

The number of sporadic cases is 12 here, but only four of them gave purely periodic solutions to (3). These were

Table 5.3

M	r	eigenvector
(5, 3, 2, 2)	$-\frac{3}{4}$	$\begin{pmatrix} 5 \\ 3 \end{pmatrix}$
(7, 4, 5, 4)	$-\frac{2}{7}$	$\begin{pmatrix} 57 \\ 34 \end{pmatrix}$
(9, 6, 5, 2)	$-\frac{151}{370}$	$\begin{pmatrix} 19849 \\ 11971 \end{pmatrix}$ (!)
(9, 2, 5, 6)	$-\frac{151}{370}$	$\begin{pmatrix} 20343 \\ 12587 \end{pmatrix}$ (!)

We note that (5, 2, 2, 3) and (5, 3, 2, 2) are reversals of each other but only one of them yields a periodic solution. This was the only instance we found where a sequence leads to a solution to (3) but its reversal does not. The infinite families with 4 B's are:

Table 5.4

M	r	eigenvector
(2n+2, 2n, 2n+2, 2n)	$-\frac{F_{2n+1}}{F_{2n+3}}$	$\begin{pmatrix} F_{2n+2} \\ F_{2n+1} \end{pmatrix}$
(2n+4, 2n, 2n+4, 2n)	$-\left(\frac{F_{n+1}}{F_{n+2}}\right)^2$	$(-1)^{n+1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2F_{n+1}\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix}$
(2n+4, 2n, 2n+4, 2n)	$-\left(\frac{L_{n+1}}{L_{n+2}}\right)^2$	$5(-1)^{n}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2L_{n+1}\begin{pmatrix} L_{n+2} \\ L_{n+1} \end{pmatrix}$

These infinite families all have the form $M = N^2$ for an appropriate N. The eigenvector for M is an eigenvector for N with an eigenvalue of -1. The first line is based on the note about $(A^nB)^2$ from Table 5.1. For the second line,

$$(1, 1)\mathbf{v} = 2((-1)^{n+1} + F_{n+1}F_{n+3}) = 2F_{n+2}^2,$$

or twice the denominator of r. To verify condition (c) of Theorem 4.1, we have

$$B\mathbf{v} = (-1)^{n+1}\begin{pmatrix} 2x \\ 1 \end{pmatrix} + 2F_{n+1}\begin{pmatrix} xF_{n+3} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} 2x(F_{n+3}F_{n+1} + (-1)^{n+1}) \\ (-1)^{n+1} + 2F_{n+2}F_{n+1} \end{pmatrix} = \begin{pmatrix} -2F_{n+1}^2 \\ (-1)^{n+1} + 2F_{n+2}F_{n+1} \end{pmatrix},$$

so $u_0 = (-1)^{n+1} + 2F_{n+2}F_{n+1} - 2F_{n+1}^2 = (-1)^{n+1} + 2F_{n+1}F_n$. Since

$q = F_{n+2}^2 = (F_{n+1} + F_n)^2 = F_{n+1}^2 + F_n^2 + 2F_{n+1}F_n,\ \ 0 < u_0 < q.$

If we write $\quad B\mathbf{v} = (-1)^{n+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2F_{n+1}\begin{pmatrix} -F_{n+1} \\ F_{n+2} \end{pmatrix},$

then by Theorem 2.5 (c)

$$A^{2n}B\mathbf{v} = (-1)^{n+1}\begin{pmatrix} F_{2n} \\ F_{2n-1} \end{pmatrix} + 2F_{n+1}\begin{pmatrix} -F_{1-n} \\ F_{2-n} \end{pmatrix}.$$

The sum of the entries in this vector is

$$\begin{aligned}(-1)^{n+1}F_{2n+1} + 2F_{n+1}F_{-n} &= (-1)^{n+1}F_{2n+1} + 2F_{n+1}(-1)^{n+1}F_n \\ &= (-1)^{n+1}(F_{2n+1} + 2F_nF_{n+1}).\end{aligned}$$

Using $F_{2n+1} = F_{n+1}^2 + F_n^2$, this expression becomes $(-1)^{n+1}(F_{n+1}^2 + 2F_nF_{n+1} + F_n^2)$, or $(-1)^{n+1}F_{n+2}^2 = \pm\, q$. A trick to show that $\mathbf{v}$ is an eigenvector for $A^{2n+4}BA^{2n}B$ is to show that $BA^{2n}B\mathbf{v} = A^{-2n-4}\mathbf{v}$. Using the calculations above,

$$BA^{2n}B\mathbf{v} = \begin{pmatrix} (-1)^nF_{n+1}^2 \\ (-1)^{n+1}F_{2n} + 2(-1)^{n-1}F_{n+1}F_{n-2} \end{pmatrix} = (-1)^{n+1}\begin{pmatrix} -F_{n+1}^2 \\ F_{2n} + 2F_{n+1}F_{n-1} \end{pmatrix}.$$

Now $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_2 \\ F_1 \end{pmatrix}$, so by Theorem 2.5 (b), $A^{-k}\mathbf{v} = (-1)^{n+1}\begin{pmatrix} F_{2-k} \\ F_{1-k} \end{pmatrix} + 2F_{n+1}\begin{pmatrix} F_{n+2-k} \\ F_{n+1-k} \end{pmatrix}.$

We have

$$\begin{aligned} A^{-2n-4}\mathbf{v} &= (-1)^{n+1}\begin{pmatrix} F_{-2n-2} \\ F_{-2n-3} \end{pmatrix} + 2F_{n+1}\begin{pmatrix} F_{-n-2} \\ F_{-n-3} \end{pmatrix} \\ &= (-1)^{n+1}\begin{pmatrix} -F_{2n+2} \\ F_{2n+3} \end{pmatrix} + (-1)^{n+1}2F_{n+1}\begin{pmatrix} F_{n+2} \\ -F_{n+3} \end{pmatrix} \\ &= (-1)^n\begin{pmatrix} F_{2n+2} - 2F_{n+2}F_{n+1} \\ 2F_{n+3}F_{n+1} - F_{2n+3} \end{pmatrix} = (-1)^n\begin{pmatrix} -F_{n+1}^2 \\ F_{2n} + 2F_{n+1}F_{n-1} \end{pmatrix}, \end{aligned}$$

as desired. To justify the last equality, we simply checked 10 consecutive cases (but three cases would have sufficed). Finally, for condition (d) of Theorem 4.1, we let $u_k = (1, 1)A^kBA^{2n}B$. We must show that u_k is not divisible by F_{n+2}^2 if $0 < k < 2n+4$. By Theorem 2.6, the u's decrease in absolute value while they alternate, and increase in absolute value while they are monotone. Also, $u_{2n+4} = -\,2F_{n+2}^2$. Thus, we must show that u_k is never $-\,F_{n+2}^2$. It is only the large values of k that cause a problem, and we can investigate them using $-\,u_{2n+4-k} = (1, 1)A^{-k}\mathbf{v} = 2F_{n+1}F_{n+3-k} + (-1)^{n+1}F_{3-k}$. Since $F_{n+2}^2 = (F_{n+1} + F_n)^2 > 2F_{n+1}F_n$, we need only worry about $k = 1, 2$. We have

$$\begin{aligned} -\,u_{2n+3} &= 2F_{n+1}F_{n+2} + (-1)^{n+1}, \\ -\,u_{2n+2} &= 2\,F_{n+1}^2 + (-1)^{n+1} = 2F_{n+2}F_n - (-1)^{n+1}. \end{aligned}$$

Clearly, neither of these is divisible by F_{n+2}, so they are not F_{n+2}^2. In the second line, we applied Cassini's identity in the form $F_{n+1}^2 = F_{n+2}F_n - (-1)^n$.

Though not needed in the proof, we note that with a little more care, we could have shown that no u_k is 0, so the number of applications of B are as advertised. The justification for the third row of Table 5.4 is entirely analogous to the proof above, but with appropriate Lucas identities substituted for Fibonacci identities.

5.5 **More than four B's**

With 5 B's, we found no infinite families. Sporadic cases that lead to purely periodic solutions to (3) are given in the table below.

Table 5.5

M	r	eigenvector
(5, 5, 5, 0, 0)	$-\frac{1}{4}$	$\begin{pmatrix} 59 \\ 33 \end{pmatrix}$
(11, 5, 9, 9, 9)	$-\frac{19}{50}$	$\begin{pmatrix} 2627 \\ 1623 \end{pmatrix}$
(11, 9, 9, 9, 5)	$-\frac{19}{50}$	$\begin{pmatrix} 27 \\ 25 \end{pmatrix}$

We only looked at a small sample of products with six B's, due to the combinatorial explosion. The following sporadic cases lead to solutions to (3):

Table 5.6

M	r	eigenvector
(5, 0, 5, 0, 2, 2)	$-\frac{1}{10}$	$\begin{pmatrix} 211 \\ 129 \end{pmatrix}$
(5, 0, 5, 2, 2, 0)	$-\frac{1}{10}$	$\begin{pmatrix} 453 \\ 277 \end{pmatrix}$
(5, 0, 2, 5, 0, 2)	$-\frac{1}{4}$	$\begin{pmatrix} 5 \\ 3 \end{pmatrix}$
(5, 2, 0, 5, 2, 0)	$-\frac{1}{4}$	$\begin{pmatrix} 13 \\ 7 \end{pmatrix}$
(8, 4, 7, 8, 4, 5)	$-\frac{64}{147}$	$\begin{pmatrix} 5211121 \\ 3279305 \end{pmatrix}$
(8, 7, 4, 8, 5, 4)	$-\frac{64}{147}$	$\begin{pmatrix} 486557 \\ 295189 \end{pmatrix}$

We found four related infinite families that appear to provide solutions to (3).

Table 5.7

M	r	eigenvector
(4n+2, 4n, 4n+1, 4n+2, 4n, 4n+1)	$-\frac{a_n}{b_n}$	$\begin{pmatrix} F_{4n+1}F_{2n+2} \\ F_{4n+2}F_{2n} \end{pmatrix}$
(4n+2, 4n+1, 4n, 4n+2, 4n+1, 4n)	$-\frac{a_n}{b_n}$	?
(4n+4, 4n+2, 4n+3, 4n+4, 4n+2, 4n+3)	$-\frac{c_n}{d_n}$	$\begin{pmatrix} F_{4n+3}L_{2n+3} \\ F_{4n+4}L_{2n+1} \end{pmatrix}$
(4n+4, 4n+3, 4n+2, 4n+4, 4n+3, 4n+2)	$-\frac{c_n}{d_n}$	?

Where $\quad a_n = F_{4n+1}F_{2n-1} + F_{4n}F_{2n+1},\ \ b_n = F_{4n+2}F_{2n} + F_{4n+1}F_{2n+2}$

and $\quad c_n = F_{4n+4}L_{2n} + F_{4n+2}L_{2n+1},\ \ d_n = F_{4n+4}L_{2n+1} + F_{4n+3}L_{2n+3}.$

As in the case of three B's, we could not find the general pattern of the eigenvectors in the second and fourth rows. Those eigenvectors did provide solutions to

(3) in all the cases we checked. We sketch the justification for the first and third rows in the next section. We close this section with a summary table of all of the infinite families we have found. In each case, we list the value of r, followed by one eigenvector. As usual, given an eigenvector $\begin{pmatrix} u \\ v \end{pmatrix}$, the initial conditions for (3) would be $a_0 = v$, $a_1 = u$.

Table 5.8

Value of r	Eigenvector
$-\dfrac{F_{2n} - 1}{F_{2n+2} + 1}$	$\begin{pmatrix} F_{2n+1} \\ F_{2n} + 1 \end{pmatrix}$
$-\dfrac{F_{2n} + 1}{F_{2n+2} - 1}$	$\begin{pmatrix} F_{2n+1} \\ F_{2n} - 1 \end{pmatrix}$
$-\dfrac{F_n}{F_{n+2}}$	$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$
$-\dfrac{L_n}{L_{n+2}}$	$\begin{pmatrix} L_{n+1} \\ L_n \end{pmatrix}$
$-\left(\dfrac{F_{n+1}}{F_{n+2}}\right)^2$	$(-1)^{n+1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2F_{n+1}\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix}$
$-\left(\dfrac{L_{n+1}}{L_{n+2}}\right)^2$	$5(-1)^n\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2L_{n+1}\begin{pmatrix} L_{n+2} \\ L_{n+1} \end{pmatrix}$
$-\dfrac{F_{4n+1}L_{2n-1} + F_{4n}L_{2n+1}}{F_{4n+2}L_{2n} + F_{4n+1}L_{2n+2}}$	$\begin{pmatrix} F_{4n+1}L_{2n+2} \\ F_{4n+2}L_{2n} \end{pmatrix}$
$-\dfrac{F_{4n+1}F_{2n-1} + F_{4n}F_{2n+1}}{F_{4n+2}F_{2n} + F_{4n+1}F_{2n+2}}$	$\begin{pmatrix} F_{4n+1}F_{2n+2} \\ F_{4n+2}F_{2n} \end{pmatrix}$
$-\dfrac{F_{4n+4}F_{2n} + F_{4n+2}F_{2n+1}}{F_{4n+4}F_{2n+1} + F_{4n+3}F_{2n+3}}$	$\begin{pmatrix} F_{4n+3}F_{2n+3} \\ F_{4n+4}F_{2n+1} \end{pmatrix}$
$-\dfrac{F_{4n+4}L_{2n} + F_{4n+2}L_{2n+1}}{F_{4n+4}L_{2n+1} + F_{4n+3}L_{2n+3}}$	$\begin{pmatrix} F_{4n+3}L_{2n+3} \\ F_{4n+4}L_{2n+1} \end{pmatrix}$

6. The infinite families (4n+2, 4n, 4n+1) and (4n+4, 4n+2, 4n+3).

In this section we prove that there is an infinite family of solutions to (3) associated with the product $A^{4n+2}BA^{4n}BA^{4n+1}B$, and sketch the proofs for the families $A^{4n+4}BA^{4n+2}BA^{4n+3}B$, $(A^{4n+2}BA^{4n}BA^{4n+1}B)^2$ and $(A^{4n+4}BA^{4n+2}BA^{4n+3}B)^2$. We post the following summary, noting the beautiful symmetry between F and L:

Table 6.1

Form	Value of r	Eigenvector
$A^{4n+2}BA^{4n}BA^{4n+1}B$	$-\dfrac{F_{4n+1}L_{2n-1} + F_{4n}L_{2n+1}}{F_{4n+2}L_{2n} + F_{4n+1}L_{2n+2}}$	$\begin{pmatrix} F_{4n+1}L_{2n+2} \\ F_{4n+2}L_{2n} \end{pmatrix}$
$(A^{4n+2}BA^{4n}BA^{4n+1}B)^2$	$-\dfrac{F_{4n+1}F_{2n-1} + F_{4n}F_{2n+1}}{F_{4n+2}F_{2n} + F_{4n+1}F_{2n+2}}$	$\begin{pmatrix} F_{4n+1}F_{2n+2} \\ F_{4n+2}F_{2n} \end{pmatrix}$
$A^{4n+4}BA^{4n+2}BA^{4n+3}B$	$-\dfrac{F_{4n+4}F_{2n} + F_{4n+2}F_{2n+1}}{F_{4n+4}F_{2n+1} + F_{4n+3}F_{2n+3}}$	$\begin{pmatrix} F_{4n+3}F_{2n+3} \\ F_{4n+4}F_{2n+1} \end{pmatrix}$
$(A^{4n+4}BA^{4n+2}BA^{4n+3}B)^2$	$-\dfrac{F_{4n+4}L_{2n} + F_{4n+2}L_{2n+1}}{F_{4n+4}L_{2n+1} + F_{4n+3}L_{2n+3}}$	$\begin{pmatrix} F_{4n+3}L_{2n+3} \\ F_{4n+4}L_{2n+1} \end{pmatrix}$

Theorem 6.2. Each row of Table 6.1 corresponds to an infinite family of periodic solutions to (3).

For example, if $a_0 = F_{4n+2}F_{2n}$, $a_1 = F_{4n+1}F_{2n+2}$, and $r = -\dfrac{F_{4n+1}F_{2n-1} + F_{4n}F_{2n+1}}{F_{4n+2}F_{2n} + F_{4n+1}F_{2n+2}}$, then $\{a_n\}$ is a periodic solution to (3) with period 2(12n + 6) or 24n + 12. To prove Theorem 6.2, we first introduce the following notation: We suppress the index n and refer to the eigenvectors in these rows as v_1, v_2, v_3, v_4. Similarly, let the values of r be r_1, r_2, r_3, r_4, and let $M_1 = A^{4n+2}BA^{4n}BA^{4n+1}B$, using r_1, $M_2 = A^{4n+2}BA^{4n}BA^{4n+1}B$ using r_2, $M_3 = A^{4n+4}BA^{4n+2}BA^{4n+3}B$ using r_3 and $M_4 = A^{4n+4}BA^{4n+2}BA^{4n+3}B$ using r_4.

The proof of Theorem 6.2 will be to show that the conditions of Theorem 4.1 are satisfied. To do that, we must prove that the v's are eigenvectors of their appropriate matrices In particular, we need

(7) $$M_1v_1 = v_1, \quad M_2v_2 = -\,v_2, \quad M_3v_3 = v_3, \quad M_4v_4 = -\,v_4.$$

Moreover, we show that certain partial products, M_i of M have the property that

$$(1, 1)M_i\mathbf{v} = \pm\, q.$$

As will be seen below, the various **v**'s of Theorem 4.1 will have entries of the same sign, so we need not worry about condition (c). Conditition (d) will not apply as our eigenvectors have the property that $(1, 1)\mathbf{v} = q$.

We proceed in stages as follows:

Lemma 6.3. Using the v's as above, and the appropriate r−values in B, we have

a. $$A^{4n+1}Bv_1 = v_1 + L_{2n+1}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

b. $$A^{4n+1}Bv_2 = -\,v_2 + F_{2n+1}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

c. $$A^{4n+3}Bv_3 = v_3 + F_{2n+2}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

d. $$A^{4n+3}Bv_4 = -\,v_4 - L_{2n+2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Proof: For (a), if we denote v_1 by$\begin{pmatrix} a \\ b \end{pmatrix}$, then $(a + b)r$ is in integer, call it − c. We have

$$A^{4n+1}Bv_1 = \begin{pmatrix} F_{4n+2} & F_{4n+1} \\ F_{4n+1} & F_{4n} \end{pmatrix}\begin{pmatrix} r & r \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} aF_{4n+1} - cF_{4n+2} \\ aF_{4n} - cF_{4n+1} \end{pmatrix},$$

So we need

(8) $$\begin{aligned} aF_{4n+1} - cF_{4n+2} &= a + L_{2n+1}, \\ aF_{4n} - cF_{4n+1} &= b - L_{2n+1}. \end{aligned}$$

With $a = F_{4n+1}L_{2n+2}$, $b = F_{4n+2}L_{2n}$, $c = F_{4n+1}L_{2n-1} + F_{4n}L_{2n+1}$,

$$\begin{aligned} aF_{4n+1} - cF_{4n+2} &= F^2_{4n+1}L_{2n+2} - F_{4n+2}F_{4n+1}L_{2n-1} - F_{4n+2}F_{4n}L_{2n-1} \\ &= F^2_{4n+1}L_{2n+2} - F_{4n+2}F_{4n+1}L_{2n-1} - (F^2_{4n+1} - 1)L_{2n-1}, \end{aligned}$$

by Cassini's identity. Thus, we must show that

$$F^2_{4n+1}L_{2n+2} - F_{4n+2}F_{4n+1}L_{2n-1} - F^2_{4n+1}L_{2n+1} = F_{4n+1}L_{2n+2},$$

or equivalently,

(9) $$F_{4n+1}L_{2n} - F_{4n+2}L_{2n-1} = L_{2n+2}.$$

The second line of (8) follows a similar pattern: We have

$$\begin{aligned} aF_{4n} - cF_{4n+1} &= F_{4n+1}F_{4n}L_{2n+2} - F_{4n+1}^2L_{2n-1} - F_{4n+1}F_{4n}L_{2n+1} \\ &= F_{4n+1}F_{4n}L_{2n} - F_{4n+1}^2L_{2n-1} \\ &= F_{4n+2}F_{4n+1}L_{2n} - F_{4n+1}^2L_{2n} - F_{4n+1}^2L_{2n-1} \\ &= F_{4n+2}F_{4n+1}L_{2n} - F_{4n+1}^2L_{2n+1} \\ &= F_{4n+2}F_{4n+1}L_{2n} - F_{4n+2}F_{4n}L_{2n+1} - L_{2n+1}. \end{aligned}$$

To get this, we used $F_{4n} = F_{4n+2} - F_{4n+1}$ in the third line and Cassini's identity to get the last line. We want $aF_{4n} - cF_{4n+1} = b - L_{2n+1}$, and this will follow if we can show that

$$F_{4n+1}L_{2n} - F_{4n}L_{2n+1} = L_{2n}. \tag{10}$$

The simplest way to verify (9) and (10) is to note that each side satisfies the same recurrence of degree at most 7 (the characteristic polynomial has at most α^6, α^4, α^2, 1, β^2, β^4, and β^6 as zeros), so checking 7 values of n suffices (we checked n = 0, …, 10).

Obviously, (b), (c), (d) are similar. In each case, a result similar to (8) must be established, and this result is equivalent to two Fibonacci/Lucas identities. For (b), the identities are:

$$F_{4n+1}F_{2n} - F_{4n+2}F_{2n-1} = -F_{2n+2},$$
$$F_{4n+1}F_{2n} - F_{4n}F_{2n+1} = -F_{2n}.$$

For (c), we need

$$F_{4n+3}F_{2n+1} - F_{4n+4}F_{2n} = F_{2n+3},$$
$$F_{4n+3}F_{2n+1} - F_{4n+2}F_{2n+2} = F_{2n+1},$$

and for (d), the required identities are

$$F_{4n+3}L_{2n+1} - F_{4n+4}L_{2n} = -L_{2n+3},$$
$$F_{4n+3}L_{2n+1} - F_{4n+2}L_{2n+2} = -L_{2n+1}.$$

Again, each of these holds for at least seven values of n, so they are identities. □

Next, we take the results of Lemma 6.3 and multiply by an appropriate A^kB.

Lemma 6.4. Using the v's as above, and the appropriate r−values in B, we have

a. $A^{4n}BA^{4n+1}Bv_1 = v_1 - L_{2n}\begin{pmatrix}1\\-1\end{pmatrix}$,

b. $A^{4n}BA^{4n+1}Bv_2 = v_2 + F_{2n}\begin{pmatrix}1\\-1\end{pmatrix}$,

c. $A^{4n+2}BA^{4n+3}Bv_3 = v_3 - F_{2n+1}\begin{pmatrix}1\\-1\end{pmatrix}$,

d. $A^{4n+2}BA^{4n+3}Bv_4 = v_4 + L_{2n+1}\begin{pmatrix}1\\-1\end{pmatrix}$.

Proof. Consider the statement in part (a). Using Lemma 6.3 (a) we have

$$\begin{aligned}A^{4n}BA^{4n+1}Bv_1 &= A^{4n}B\left(v_1 + L_{2n+1}\begin{pmatrix}1\\-1\end{pmatrix}\right)\\ &= A^{-1}A^{4n+1}B\left(v_1 + L_{2n+1}\begin{pmatrix}1\\-1\end{pmatrix}\right)\\ &= A^{-1}\left(v_1 + L_{2n+1}\begin{pmatrix}1\\-1\end{pmatrix}\right) + L_{2n+1}A^{4n}B\begin{pmatrix}1\\-1\end{pmatrix}.\end{aligned}$$

Using the notation of Lemma 6.3, we require the following:

$$b - L_{2n+1} + F_{4n}L_{2n+1} = a - L_{2n},$$

$$a - b + 2L_{2n+1} + F_{4n-1}L_{2n+1} = b + L_{2n}$$

As in Lemma 6.3, we need only verify these for seven values of n. Parts (b), (c), (d) follow the same pattern. □

Lemma 6.5 The equations in (7) hold.

Proof. The proof is essentially the same as above. We provide one sample calculation. To show that $A^{4n+4}BA^{4n+2}BA^{4n+3}Bv_4 = -v_4$,
by Lemma 6.4 (d) we have

$$\begin{aligned}A^{4n+4}BA^{4n+2}BA^{4n+3}Bv_4 &= A^{4n+4}B\left(v_4 + L_{2n+1}\begin{pmatrix}1\\-1\end{pmatrix}\right)\\ &= A\left(-v_4 - L_{2n+2}\begin{pmatrix}1\\-1\end{pmatrix}\right) + L_{2n+1}A^{4n+4}\begin{pmatrix}0\\1\end{pmatrix}.\end{aligned}$$

Letting $v_4 = \begin{pmatrix}a\\b\end{pmatrix}$, we must verify that

$$-\begin{pmatrix}a+b\\a\end{pmatrix} - L_{2n+2}\begin{pmatrix}0\\1\end{pmatrix} + L_{2n+1}\begin{pmatrix}F_{4n+4}\\F_{4n+3}\end{pmatrix} = -\begin{pmatrix}a\\b\end{pmatrix},$$

or $$b = F_{4n+4}L_{2n+1},$$

$$a - b = F_{4n+3}L_{2n+1} - L_{2n+2}.$$

The first is true by definition, the second follows by checking seven values of n. □

Proof of Theorem 6.2. The lemmas show that the conditions of Theorem 4.1 are met, so the given vectors do provide families of periodic solutions to (3). We note that a careful analysis will show that the proper number of B's is used at each stage. That is, when using Theorem 4.1, no u_k is 0. We simply sketch the proof of this. First, if some u_k was zero, by Theorem 2.6 (c), the u's would have to all have the form $u_k = cF_{k-j}$ for some integers c and j. However, the entries in v are relatively prime and this can be used to show $c = \pm 1$. This means that consecutive u's must be consecutive Fibonacci numbers. Since this is not the case, no u_k can be 0. □

7. **Concluding Remarks**.

Missing from this paper are any strong results about the non existence of solutions to (3). Whereas we have shown that the only possible positive values of r for which (3) admits periodic solutions are reciprocals of integers, we suspect the much stronger result that $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$ are the only positive values that admit periodic solutions, and that the only possible periodic solutions for these with relatively prime initial conditions are the known ones. We have verified that any other positive r must have a period longer that 24.

Among the negative values of r, again, we have not classified the periodic solutions for any value of r, but only exhibited a periodic solution for the r–values we have found. We can not rule out any value of r with $-1 < r < 0$ which we have not discovered. We have found all values of r having periodic solutions of length at most 24, and we have found all values of r with much larger periods if we restrict the number of B's involved.

Even in the restricted search for periodic solutions to (3) which use a prescribed number of steps of the form $a_n = r(a_{n-1} + a_{n-2})$, we can say very little. We have classified all such solutions in which there is only one of these steps. If there are two, we are fairly confident that they are all characterized in Table 5.1 because we have explicitly calculated all r for which A^mBA^nB has 1 as an eigenvalue, with $0 \le m, n \le 100$. It might not be too hard to prove that Table 5.1 is complete. If $M = A^mBA^nB$, then

(11) $$f_M(x) = (F_{m+2}F_{n+2} - (-1)^{m+n})\, x^2 + 2F_{m+1}F_{n+1}\, x + F_mF_n - 1.$$

The discriminant of this polynomial is

(12) $$(F_{m+2} + (-1)^m F_n)(F_{n+2} + (-1)^n F_m),$$

so rational solutions occur only if this is a square. If we restrict to the case where $m \geq n$, (12) was only found to be a square for $(m, n) = (1, 0)$, $(2, 1)$, or for the infinite families $m = n$, and $m = n + 2$, with n odd. We do not know how to prove that these are the only cases, but see [4] where it is proved that there are only finitely many Fibonacci numbers or Lucas numbers that are squares.

The only infinite families of M's we found with more than three B's all have the form $M = N^2$. It is possible that we have found all such infinite families, though we are not confident enough to state such a thing as a conjecture. We at least looked at the case $M = N^2$, where N has four B's, and with as many as 16 A's between B's. We found no obvious infinite families leading to solutions to (3). We tried a similar search for N with five B's, again without success.

We would like to have formulas for the missing eigenvectors in Tables 5.2 and 5.7. We would also like to see proofs that these eigenvectors give periodic solutions to (3). This could be difficult since Theorem 4.1 will not apply as the entries in the eigenvectors are too big. For example, both $A^8BA^7BA^6B$ and $A^8BA^6BA^7B$ have eigenvectors when $r = -\frac{37}{107}$; the eigenvector for the latter one is $\begin{pmatrix} 65 \\ 42 \end{pmatrix}$, but for the former, it is $\begin{pmatrix} 3157 \\ 1979 \end{pmatrix}$.

Finally, we mention that the infinite families with three B's (or infinite families with 6 B's in which M is a square) have a consequence for Fibonacci numbers. If $M = A^{2n+2}BA^{2n+1}BA^{2n}B$, then

(13) $$f_M(x) = (F_{2n+4}F_{2n+3}F_{2n+2} - 1)\, x^3 + (F_{2n+3}^3 + F_{2n+2}^2F_{2n+1})\, x^2 \\ + (F_{2n+3}F_{2n+2}^2 + F_{2n+1}^3)\, x + F_{2n+2}F_{2n+1}F_{2n} - 1.$$

This polynomial always has a linear factor since it has a rational zero. From the values of r given in Table 6.1, we are led to the following identities:

$$F_{4n+2}F_{4n+1}F_{4n} - 1 = (F_{4n}F_{2n-1} + F_{4n-1}F_{2n+1})(F_{4n+1}L_{2n-1} + F_{4n}L_{2n+1}),$$
$$F_{4n+4}F_{4n+3}F_{4n+2} - 1 = (F_{4n+2}L_{2n} + F_{4n}L_{2n+2})(F_{4n+4}F_{2n} + F_{4n+2}F_{2n+1}). \quad (14)$$

Similarly, from the fact that $M = (A^{2n+2}BA^{2n+1}BA^{2n}B)^2$ has values of r leading to periodic solutions, and from their form in Table 6.1, we have the identities

$$F_{4n+2}F_{4n+1}F_{4n} + 1 = (F_{4n}L_{2n-1} + F_{4n-1}L_{2n+1})(F_{4n+1}F_{2n-1} + F_{4n}F_{2n+1}),$$
$$F_{4n+4}F_{4n+3}F_{4n+2} + 1 = (F_{4n+2}F_{2n} + F_{4n}F_{2n+2})(F_{4n+4}L_{2n} + F_{4n+2}L_{2n+1}). \quad (15)$$

We had great difficulty establishing these identities, though, of course, they can be verified by checking a finite number of cases. On the other hand, for odd n, $F_{n+2}F_{n+1}F_n \pm 1$ does not have a nice factorization as witnessed by the fact that $F_{17}F_{16}F_{15} - 1$ and $F_{15}F_{14}F_{13} + 1$ are prime.

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